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Wave propagation in non-Gaussian random media

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Abstract

We develop a compact perturbative series for acoustic wave propagation in a medium with a non-Gaussian stochastic speed of sound. We use Martin–Siggia and Rose auxiliary field techniques to render the classical wave propagation problem into a 'quantum' field theory one, and then frame this problem within the so-called Schwinger–Keldysh of closed time-path (CTP) formalism. Variation of the so-called two-particle irreducible (2PI) effective action (EA), whose arguments are both the mean fields and the irreducible two point correlations, yields the Schwinger–Dyson and the Bethe–Salpeter equations. We work out the loop expansion of the 2PI CTP EA and show that, in the paradigmatic problem of overlapping spherical intrusions in an otherwise homogeneous medium, non-Gaussian corrections might be much larger than Gaussian ones at the same order of loops.

Keywords: waves, random media, field theory methods

1. Introduction

Wave propagation in a heterogeneous medium [1–5] is a problem with an enormous range of applications, from remote sensing to Anderson localization [6–8]. In many applications, the medium may be modeled as having a point-dependent, partially random speed of sound (in this paper we shall discuss the scalar wave equation only; the generalization of our work to the electromagnetic case is straightforward). There are two main objects to be computed: first, the self-energy or mass operator, which is essentially the inverse square of the speed of propagation of the wave in the medium and thus is necessary to compute the coherent part of the field, and second, the intensity operator, which is necessary to compute field fluctuations and thereby the incoherent scattering. We give precise definitions of both of these objects below. These objects are constrained by the symmetries of the theory, namely, flux conservation (in the scalar case; in the electromagnetic case, the relevant conserved quantity is energy) and reciprocity [9–11]. Flux conservation implies a constraint that the mass and intensity operator must satisfy, the so-called Ward identity [12–14].

In $d = 3$ dimensions, the problem is already too hard for a closed form solution, and one must resort to perturbation theory, the small parameter being associated to the strength of random fluctuations in the speed of sound. Field theory methods have been extensively applied in the development of suitable perturbative schemes [15–18]. In the most straightforward formulation, the field equation is first transformed into an integral equation which is then solved iteratively. One obtains in this way a linear mean field equation for the coherent field, a Schwinger–Dyson equation for the Green function associated with the mean field equation (which measures the response of the mean field to an external source) and the Bethe–Salpeter equation for the two-point correlation of incoherent field fluctuations [19–26]. One could write down these equations for a given realization of the stochastic speed of sound. They will contain kernels composed of products of the random part of the inverse square speed of sound at various points. Replacing those products by their statistical averages, one finally finds the mass and intensity operators, and therefore the final equations for the Green function and the incoherent field correlations. The first nontrivial approximation yields the so-called nonlinear approximation for the mass operator and the ladder approximation for the intensity operator (see below). These approximations are consistent with the Ward identity but not with reciprocity. To restore reciprocity (and also to account for other phenomena such as coherent backscattering and Anderson localization) it is necessary to move to higher orders in perturbation theory [27–37] (see also [38–41]).

In doing so, one encounters expressions involving products of the random inverse square speed of sound at three or more different points. The usual perturbative approach computes these higher correlations as if the random fluctuations were Gaussian, namely, higher correlations are factorized into binary correlations according to Wick's theorem. The only information kept about the medium is the two-point correlation of fluctuations. This is unsatisfactory, as many relevant applications involve non-Gaussian fluctuations [42]; for example, the fluctuating speed of sound which is obtained by randomly distributing bubbles into an otherwise homogeneous medium [43, 44] is generally non-Gaussian, as we will show below.

Hori and co-workers have developed a statistical theory which may deal with non-Gaussian fluctuations [45–51]. However, to the best of our knowledge, their work is oriented mostly to computing the mass operator. Our aim is to develop a formalism which is more suitable to the computation of both coherent and incoherent scattering, and in the process to present perturbation theory in a way which we believe is more compact and therefore easier to pursue to higher orders. As a particularly compelling feature, the perturbative scheme to be presented below has flux conservation built in order-by-order in perturbation theory.

To achieve our goals we shall use Martin, Siggia and Rose auxiliary field techniques [52–56] to render the classical wave propagation problem into a 'quantum' field theory one, and then frame this problem within the so-called Schwinger–Keldysh or closed time-path (CTP) formalism [57–62]. By coupling the different fields and their binary products to fictitious sources, one obtains a CTP-generating function for the field correlations of any order. The Legendre transform of the CTP generating function yields the so-called two-particle irreducible (2PI) effective action (EA), whose arguments are both the mean fields and the correlations [63–66]. Variation of the EA yields the Schwinger–Dyson and the Bethe–Salpeter equations, whereby it is straightforward to identify the mass and intensity operators.

The key to the power of the method is that the seeming complexity of dealing with a larger number of fields (auxiliary and ghost [67, 68] fields on top of the physical ones) is overridden by the fact that the 2PI EA is built from Feynman graphs with no external legs and which are 2PI, meaning that they remain connected after cutting any two internal legs. The number of these graphs at any finite order in perturbation theory is small enough that the computational effort stays manageable, and their structure is so tightly constrained that it is possible to provide proofs of key features, such as flux conservation, at any order [69].

The rest of the paper is organized as follows. In section 2, we give a brief overview of the perturbative theory based on a direct iteration of the field equations. This allows us to give precise definitions of the key concepts, such as Green functions, two-point correlations, Schwinger–Dyson and Bethe–Salpeter equations and the Ward identity. In section 3, we present our approach, based on the Martin–Siggia–Rose and CTP formalisms. Here, we work *as if* we had a closed-form expression for the 2PI EA. In section 4, we discuss the loop expansion of the 2PI EA. Then, as an application, we discuss the sensibility of the mass and intensity operators to the third-order correlations in the inverse square speed of sound in the case of a medium composed of spherical, interpenetrating bubbles randomly dispersed into a homogeneous matrix.

We conclude with some brief, final remarks.

2. The direct approach

We consider a complex scalar field Φ obeying a wave equation

$$\left[\boldsymbol{\Delta} + \epsilon(x)\omega^2\right]\Phi = -j, \tag{1}$$

where $\epsilon = 1/c^2$, c being the speed of sound in the medium. In turn

$$\epsilon(x) = \bar{\epsilon}(x) + \varepsilon(x) \tag{2}$$

ε is a stochastic (real) variable with zero mean and two point correlations $\langle\varepsilon(x)\varepsilon(x')\rangle = C(x, x')$. We consequently split $\Phi = \phi + \varphi$, where $\langle\varphi\rangle = 0$.

There are a number of properties of the theory which can be derived directly from the equations of motion, independently of any perturbative scheme. We begin by reviewing some of them.

Observe that for a given realization of the noise, equation (1) is linear and admits a Green function

$$\left[\boldsymbol{\Delta} + \epsilon(x)\omega^2\right]G_\epsilon(x, y) = -\delta(x - y). \tag{3}$$

The background field ϕ does not appear in this equation, and so G_ϵ is background-field independent. The solution to equation (1) is

$$\Phi(x) = \int \mathrm{d}y\; G_\epsilon(x, y)j(y). \tag{4}$$

Taking the expectation value of equation (4), we get

$$\phi(x) = \int \mathrm{d}y\; G(x, y)j(y), \tag{5}$$

where

$$G(x, y) = \left\langle G_\epsilon(x, y)\right\rangle \tag{6}$$

is independent of ϕ. Taking the expectation value of equation (3), we get

$$\left[\boldsymbol{\Delta} + \bar{\epsilon}(x)\omega^2\right]G(x, y) = -\delta(x - y) - \omega^2\left\langle \varepsilon(x)G_\epsilon(x, y)\right\rangle. \tag{7}$$

We *define* the self-energy $\Sigma(x, y)$ from the identity

$$\omega^2 \left\langle \varepsilon(x) G_\epsilon(x, y) \right\rangle = \int \mathrm{d}z\ \Sigma(x, z) G(z, y). \tag{8}$$

Therefore,

$$\left[\boldsymbol{\Delta} + \bar{\epsilon}(x)\omega^2\right] G(x, y) + \int \mathrm{d}z\ \Sigma(x, z) G(z, y) = -\delta(x - y) \tag{9}$$

and also the mass operator $Q(x, y)$ from

$$\omega^2 \langle \varepsilon(x) \varphi(x) \rangle = \int \mathrm{d}z\ Q(x, z) \phi(z). \tag{10}$$

Therefore, the mean field equation reads

$$\left[\boldsymbol{\Delta} + \bar{\epsilon}(x)\omega^2\right] \phi(x) + \int \mathrm{d}z\ Q(x, z) \phi(z) = -j(x). \tag{11}$$

On the other hand, acting with $\boldsymbol{\Delta} + \bar{\epsilon}(x)\omega^2 + \Sigma$ on both sides of equation (5) we conclude that

$$\left[\boldsymbol{\Delta} + \bar{\epsilon}(x)\omega^2\right] \phi(x) + \int \mathrm{d}z\ \Sigma(x, z) \phi(z) = -j(x). \tag{12}$$

Therefore, $Q = \Sigma$. Provided natural boundary conditions are chosen, G_ϵ is symmetric. This is the so-called reciprocity property.

From the field Φ, we could construct the current

$$J_\epsilon = (-\mathrm{i}) \left[\Phi^* \nabla \Phi - \Phi \nabla \Phi^* \right], \tag{13}$$

which obeys

$$\nabla J_\epsilon = (-\mathrm{i}) \left[-\Phi^* j + \Phi j^* \right]. \tag{14}$$

Therefore, the expectation value

$$J = \left\langle J_\epsilon \right\rangle \tag{15}$$

conserves charge in the mean, namely,

$$\nabla J = (-\mathrm{i}) \left[-j \phi^* + j^* \phi \right]. \tag{16}$$

To compute the current, we need the expectation value

$$\langle \Phi^*(x') \Phi(x) \rangle = \phi^*(x') \phi(x) + \langle \varphi^*(x') \varphi(x) \rangle. \tag{17}$$

We define the 'intensity' operator from the identity (the so-called Bethe–Salpeter equation)

$$\langle \varphi^*(x') \varphi(x) \rangle = \int \mathrm{d}z \mathrm{d}z' \mathrm{d}y \mathrm{d}y'\ G(x, z) G^*(x', z') I[z, y; z', y'] \langle \Phi^*(y') \Phi(y) \rangle. \tag{18}$$

The average of the conservation law over space–time yields the Ward identity

$$Q(y, y')^* - Q(y', y) - \int \mathrm{d}z \mathrm{d}z' \left[G^*(z, z') - G(z', z) \right] I[z, y; z', y'] = 0. \tag{19}$$

The simplest expression for the mass operator is given by the so-called nonlinear approximation

$$\Sigma_{\mathrm{nonlin}}(x, x') = Q_{\mathrm{nonlin}}(x, x') = \omega^4 C(x, x') G(x, x'). \tag{20}$$

Once this is accepted, the ladder approximation

$$I_{\text{ladder}}[x, y; x', y'] = \omega^4 C(x, x')\delta(x - y)\delta(x' - y') \tag{21}$$

provides the simplest solution to the Ward identity.

We could also write

$$\langle \Phi^*(x')\Phi(x)\rangle = \int \mathrm{d}y\mathrm{d}y' \; U[x, y; x', y']j(y)j^*(y'). \tag{22}$$

Then,

$$U[x, y; x', y'] = \langle G_\epsilon(x, y)G_\epsilon^*(x', y')\rangle. \tag{23}$$

The symmetry of the propagators implies the reciprocity condition

$$U[x, y; x', y'] = U[x, y; y', x']. \tag{24}$$

On the other hand, we have the identity

$$U = \left[1 - GG^*I\right]^{-1} GG^*. \tag{25}$$

The ladder approximation, namely, the value of U which is obtained if we substitute I_{ladder} into equation (25), violates the reciprocity condition.

3. The functional approach

The functional approach begins with the observation that moments of the stochastic Φ field could be derived from a generating functional

$$\mathrm{e}^{\mathrm{i}W\left[\mathcal{J}, \mathcal{J}^*\right]} = \int D\Phi D\Phi^* \mathcal{P}\left[\Phi, \Phi^*\right]\mathrm{e}^{\mathrm{i}\left[\mathcal{J}\Phi + \mathcal{J}^*\Phi^*\right]}. \tag{26}$$

Integration is understood in the exponent. $\mathcal{P}$ is the probability density

$$\mathcal{P}\left[\Phi, \Phi^*\right] = \int D\varepsilon \mathcal{F}[\varepsilon]\delta(\Phi - \Phi[\epsilon, j])\delta\left(\Phi^* - \Phi\left[\epsilon, j^*\right]\right), \tag{27}$$

where $\mathcal{F}[\varepsilon]$ is the probability density for the multiplicative noise, and $\Phi[\epsilon, j]$ is the solution to equation (1) for a given noise realization. Observe that

$$\delta(\Phi - \Phi[\epsilon, j]) = \mathrm{Det}\left[\mathbf{D}_\epsilon\right]\delta\left(\mathbf{D}_\epsilon\Phi + j\right), \tag{28}$$

where $\mathbf{D}_\epsilon = \boldsymbol{\Delta} + \epsilon\omega^2$. We exponentiate the delta functions by adding auxiliary fields ψ and ψ^* and the determinants by adding ghost fields $(\xi, \eta, \zeta, \vartheta)$ [67, 68]. We also parameterize

$$\mathcal{F}[\varepsilon] \propto \exp\{-F[\varepsilon]\}, \tag{29}$$

$$F[\varepsilon] = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1 \ldots \mathrm{d}x_n \; f_n(x_1, \ldots, x_n)\varepsilon(x_1)\ldots\varepsilon(x_n) \tag{30}$$

f_0 is an overall normalization constant which could be absorbed into the path integration measure; f_1 is included to enforce $\langle\varepsilon\rangle = 0$. Observe that, in practice, we have traded the non-Gaussian statistics for an infinite hierarchy of interactions [70–72]; this is manageable, nevertheless, because only a finite number of interactions will be effective at any order in the loop expansion [73].

The result is that the original stochastic theory is equivalent to a quantum field theory for fields

$$X^A = \left(\varepsilon, \Phi, \psi^*, \Phi^*, \psi, \xi, \eta, \zeta, \vartheta\right) \tag{31}$$

($A = 0$–8) with classical action

$$S = \mathrm{i}F[\varepsilon] + \psi^*\left[\mathbf{D}_\epsilon \Phi + j\right] + \psi\left[\mathbf{D}_\epsilon \Phi^* + j^*\right] + \mathrm{i}\xi \mathbf{D}_\epsilon \eta + \mathrm{i}\zeta \mathbf{D}_\epsilon \vartheta. \tag{32}$$

Observe that η and ϑ have ghost charge 1, ξ and ζ have ghost charge -1, and all other fields have ghost charge 0. From now on, we shall refer to Φ, ψ^*, Φ^* and ψ as 'matter' fields, and to ξ, η, ζ and ϑ as ghost fields.

The structure of the classical action has a number of consequences for the expectation value. Foremost, observe that $W[\mathcal{J} = \mathcal{J}^* = 0] = 0$ independently of j and j^*. Therefore, we have that *the expectation value of any product of ψ^* and ψ fields vanishes*, since any such product can be obtained as some derivative of W with respect to j and j^*. With the same argument, we could show that

$$\langle \psi^* \epsilon \rangle = \langle \psi \epsilon \rangle = \langle \psi^* \varphi^* \rangle = \langle \psi \varphi \rangle = 0. \tag{33}$$

The identity

$$\int DX^A \frac{\delta S}{\delta \psi^*} \mathrm{e}^{\mathrm{i}S} = 0 \tag{34}$$

implies that

$$\mathbf{D}_0 \phi + \omega^2 \langle \varepsilon \varphi \rangle = -j, \tag{35}$$

where

$$\mathbf{D}_0 = \boldsymbol{\Delta} + \bar{\varepsilon}\omega^2. \tag{36}$$

Thus, reproducing equation (12), after we identify ϕ as the expectation value of the 'quantum' field $\Phi = \phi + \varphi$, we therefore get an interpretation of the mass operator Q, now defined through equation (10)

$$Q = \omega^2 \frac{\delta \langle \varepsilon \varphi \rangle}{\delta \phi}. \tag{37}$$

The propagator is defined from equation (5)

$$G(x, x') = \frac{\delta \phi(x)}{\delta j(x')}. \tag{38}$$

The variational derivative can be computed from the path integral to get

$$G = \frac{\delta \phi}{\delta j} = \mathrm{i}\langle \varphi \psi^* \rangle. \tag{39}$$

3.1. The 2PI EA

The most convenient way to derive the equations of motion for the mean field and propagators is through the 2PI EA. We are interested in a situation where only Φ and Φ^* develop a nontrivial expectation value. In particular, all objects with a nonzero ghost number will vanish on shell.

To obtain the 2PI EA, we first generalize the generating functional equation (26) by including sources for all fields, and also sources coupled to binary products of fields

$$\mathrm{e}^{\mathrm{i}W[\mathcal{J}_A,\mathcal{K}_{AB}]} = \int DX^A \mathrm{e}^{\mathrm{i}\left[S\left[X^A\right]+\mathcal{J}_A X^A+\frac{1}{2}X^A\mathcal{K}_{AB}X^B\right]}. \tag{40}$$

This generating functional generates the expectation values and binary correlations of the fields

$$W\overleftarrow{\frac{\delta}{\delta\mathcal{J}_A}} = \bar{X}^A,$$
$$W\overleftarrow{\frac{\delta}{\delta\mathcal{K}_{AB}}} = \frac{1}{2}\theta^{BA}\theta^B\left[\bar{X}^A\bar{X}^B + G^{AB}\right], \tag{41}$$

where $\bar{X}^A = \langle X^A\rangle$, $G^{AB} = \langle X^A X^B\rangle - \bar{X}^A\bar{X}^B$, and $\theta^B = (-1)^{q_B}$, $\theta^{BA} = (-1)^{q_B q_A}$, q_B being the ghost charge of field X^B. These bookkeeping devices are necessary to deal with matter and ghost fields on the same footing (see [57, 68]), but they can be safely ignored in a first reading. Therefore, if we define the Legendre transform [57, 68]

$$\Gamma = W - \bar{X}^A\mathcal{J}_A - \frac{1}{2}\bar{X}^A\mathcal{K}_{AB}\bar{X}^B - \frac{1}{2}\theta^{BA}\theta^B G^{AB}\mathcal{K}_{AB} \tag{42}$$

variation of the EA yields the equations of motion

$$\frac{\delta}{\delta\bar{X}^A}\Gamma = -\mathcal{J}_A - \mathcal{K}_{AB}\bar{X}^B,$$
$$\frac{\delta}{\delta G^{AB}}\Gamma = -\frac{1}{2}\theta^{BA}\theta^B\mathcal{K}_{AB}. \tag{43}$$

We make the ansatz

$$\Gamma = S\left[\bar{X}^A\right] + \frac{1}{2}\theta^{BA}\theta^B G^{AB}S_{,AB} - \frac{\mathrm{i}}{2}\ln\,\mathrm{sdet}\left[G^{AB}\right] + \Gamma_2\left[G^{AB}\right] + \mathrm{const.} \tag{44}$$

$S_{,AB}$ comes from the second variation of the action

$$S\left[\bar{X}^A + x^A\right] = S\left[\bar{X}^A\right] + S_{,A}x^A + \frac{1}{2}x^A S_{,AB}x^B + S_{\mathrm{int}}\left[x^A\right]. \tag{45}$$

In our case S_{int} is independent of the $\bar{X}^A$; this will be a large simplification in what follows. The so-called superdeterminant (sdet) is defined through the Gaussian integration formula

$$\left[\mathrm{sdet}\left[G^{AB}\right]\right]^{1/2} = \int Dx^A\ \mathrm{e}^{\frac{-1}{2}x^A\left(G^{-1}\right)^{(L)}_{AB}x^B} \tag{46}$$

we have absorbed all universal constants into the measure, and the superscript L denotes a left inverse. We have the variational formula

$$\frac{\delta}{\delta G^{AB}}\left[\mathrm{sdet}\left[G^{AB}\right]\right] = \theta^{BA}\theta^A\left[\mathrm{sdet}\left[G^{AB}\right]\right]\left(G^{-1}\right)^{(R)}_{AB}. \tag{47}$$

The superscript R reminds us that this is the right inverse of G^{AB}. Then, $\Gamma_2\left[G^{AB}\right]$ is given by

$$\mathrm{e}^{\mathrm{i}\Gamma_2} = \left[\mathrm{sdet}\left[G^{AB}\right]\right]^{-1/2}\int Dx^A\ \mathrm{e}^{\left\{\frac{-1}{2}x^A\left(G^{-1}\right)^{(L)}_{AB}x^B + \mathrm{i}\left(S_{\mathrm{int}}\left[x^A\right]+\chi_A x^A+\frac{1}{2}x^A\kappa_{AB}x^B\right)\right\}}. \tag{48}$$

The external sources enforce the constraints $\langle x^A\rangle = 0$, $\langle x^A x^B\rangle = G^{AB}$. Because of these constraints, the sum of tadpole insertions and of self-energy corrections to any given graph vanishes; this includes all graphs where the sources appear explicitly. Therefore, to compute Γ_2, we could disregard the external sources, provided at the same time we disregard all graphs which are one- or two-particle reducible, that is, that could be seen as a tadpole insertion or a self-energy correction upon some other graph. In other words, Γ_2 is the sum of all the vacuum, 2PI Feynman graphs generated from the vertices contained in $S_{\rm int}$ carrying propagators G^{AB} in their internal lines.

The equations for the propagators are

$$S_{,AB} - \mathrm{i}\left(G^{-1}\right)^{(L)}_{AB} + 2\theta^{AB}\theta^{B}\frac{\delta\Gamma_2}{\delta G^{AB}} = 0. \tag{49}$$

Now,

$$S_{,AB} = \begin{pmatrix} H_{ab} & 0 \\ 0 & M^{ij} \end{pmatrix}, \tag{50}$$

$$H_{,ab} = \begin{pmatrix} \mathrm{i}f_2 & 0 & \omega^2\phi & 0 & \omega^2\phi^* \\ 0 & 0 & \mathbf{D}_0 & 0 & 0 \\ \omega^2\phi & \mathbf{D}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_0 \\ \omega^2\phi^* & 0 & 0 & \mathbf{D}_0 & 0 \end{pmatrix}, \tag{51}$$

$$M^{ij} = \begin{pmatrix} 0 & \mathrm{i}\mathbf{D}_0 & 0 & 0 \\ -\mathrm{i}\mathbf{D}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\mathbf{D}_0 \\ 0 & 0 & -\mathrm{i}\mathbf{D}_0 & 0 \end{pmatrix}. \tag{52}$$

Let us investigate the bosonic propagators. We have

$$S_{,rt}G^{ts} = \mathrm{i}\delta_r^s - 2\frac{\delta\Gamma_2}{\delta G^{rt}}G^{ts}. \tag{53}$$

Setting $r = 1$ ($X^1 = \varphi$), we get

$$\mathbf{D}_0\langle\psi^* X^s\rangle = \mathrm{i}\delta_1^s - 2\frac{\delta\Gamma_2}{\delta\langle\varphi X^t\rangle}\langle X^t X^s\rangle. \tag{54}$$

If we further set $s = 2$ and $s = 4$, we get

$$\frac{\delta\Gamma_2}{\delta\langle\varphi\varphi\rangle} = \frac{\delta\Gamma_2}{\delta\langle\varphi\varphi^*\rangle} = 0. \tag{55}$$

Similarly, from $r = 3$

$$\mathbf{D}_0\langle\psi X^s\rangle = \mathrm{i}\delta_3^s - 2\frac{\delta\Gamma_2}{\delta\langle\varphi^* X^t\rangle}\langle X^t X^s\rangle. \tag{56}$$

Setting $s = 4$, we get

$$\frac{\delta\Gamma_2}{\delta\langle\varphi^*\varphi^*\rangle} = 0. \tag{57}$$

Now setting $s = 0$, we get

$$\frac{\delta\Gamma_2}{\delta\langle\varepsilon\varphi\rangle} = \frac{\delta\Gamma_2}{\delta\langle\varepsilon\varphi^*\rangle} = 0. \tag{58}$$

We see that equation (54) reduces to

$$\left[\mathbf{D}_0 + 2\frac{\delta \Gamma_2}{\delta\langle\varphi\psi^*\rangle}\right]\langle\psi^*\varphi\rangle = \mathrm{i}\mathbf{1}. \tag{59}$$

Let us go back to equation (53) and write the remaining equations. With $r = 0$, we get

$$\left[\mathrm{i}f_2 + 2\frac{\delta \Gamma_2}{\delta\langle\varepsilon\varepsilon\rangle}\right]\langle\varepsilon X^s\rangle + \left[\omega^2\phi + 2\frac{\delta \Gamma_2}{\delta\langle\varepsilon\psi^*\rangle}\right]\langle\psi^* X^s\rangle + \left[\omega^2\phi^* + 2\frac{\delta \Gamma_2}{\delta\langle\varepsilon\psi\rangle}\right]\langle\psi X^s\rangle = \mathrm{i}\delta_0^s. \tag{60}$$

With $s = 0$, we get

$$\left[\mathrm{i}f_2 + 2\frac{\delta \Gamma_2}{\delta\langle\varepsilon\varepsilon\rangle}\right]\langle\varepsilon\varepsilon\rangle = \mathrm{i}\mathbf{1}. \tag{61}$$

With $s = 1$

$$\langle\varepsilon\varphi\rangle = C\left[\omega^2\phi + 2\frac{\delta \Gamma_2}{\delta\langle\varepsilon\psi^*\rangle}\right]G', \tag{62}$$

where $\langle\varepsilon(x)\varepsilon(x')\rangle = C(x, x')$ and $G' = \mathrm{i}\langle\psi^*\varphi\rangle$. Now set $r = 2$

$$\left[\omega^2\phi + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varepsilon\rangle}\right]\langle\varepsilon X^s\rangle + \left[\mathbf{D}_0 + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varphi\rangle}\right]\langle\varphi X^s\rangle$$
$$+2\frac{\delta \Gamma_2}{\delta\langle\psi^*\psi^*\rangle}\langle\psi^* X^s\rangle + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varphi^*\rangle}\langle\varphi^* X^s\rangle + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\psi\rangle}\langle\psi X^s\rangle = \mathrm{i}\delta_2^s. \tag{63}$$

With $s = 2$, this leads to

$$\left[\omega^2\phi + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varepsilon\rangle}\right]\langle\varepsilon\psi^*\rangle + \left[\mathbf{D}_0 + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varphi\rangle}\right]\langle\varphi\psi^*\rangle$$
$$+2\frac{\delta \Gamma_2}{\delta\langle\psi^*\psi^*\rangle}\langle\psi^*\psi^*\rangle + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varphi^*\rangle}\langle\varphi^*\psi^*\rangle + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\psi\rangle}\langle\psi\psi^*\rangle = \mathrm{i}\mathbf{1}. \tag{64}$$

Now, $\langle\varepsilon\psi^*\rangle = \langle\psi^*\psi^*\rangle = \langle\psi\psi^*\rangle = \langle\varphi^*\psi^*\rangle = 0$, so this reduces to

$$\left[\mathbf{D}_0 + 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varphi\rangle}\right]\langle\varphi\psi^*\rangle = \mathrm{i}\mathbf{1}. \tag{65}$$

We identify $G = \mathrm{i}\langle\varphi\psi^*\rangle$ and therefore

$$\Sigma = Q = 2\frac{\delta \Gamma_2}{\delta\langle\psi^*\varphi\rangle}. \tag{66}$$

With $s \neq 2$, we get

$$\langle \varphi X^s \rangle = G\left\{\left[\omega^2\phi + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\varepsilon\rangle}\right]\langle \varepsilon X^s\rangle + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\psi^*\rangle}\langle\psi^* X^s\rangle + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\varphi^*\rangle}\langle\varphi^* X^s\rangle \right.$$
$$\left. + \ 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\psi\rangle}\langle\psi X^s\rangle\right\}. \tag{67}$$

If $s = 0$

$$\langle\varphi\varepsilon\rangle = G\left\{\left[\omega^2\phi + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\varepsilon\rangle}\right]C + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\varphi^*\rangle}\langle\varphi^*\varepsilon\rangle\right\}. \tag{68}$$

Compared with equation (62), we conclude that

$$\frac{\delta\Gamma_2}{\delta\langle\psi^*\varphi^*\rangle} = 0. \tag{69}$$

We can reach the same conclusion by setting $s = 4$. Finally, with $s = 1$ and $s = 3$, we get

$$\langle\varphi\varphi\rangle = G\left\{\left[\omega^2\phi + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\varepsilon\rangle}\right]\langle\varepsilon\varphi\rangle + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\psi^*\rangle}\langle\psi^*\varphi\rangle\right\}, \tag{70}$$

$$\langle\varphi\varphi^*\rangle = G\left\{\left[\omega^2\phi + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\varepsilon\rangle}\right]\langle\varepsilon\varphi^*\rangle + 2\frac{\delta\Gamma_2}{\delta\langle\psi^*\psi\rangle}\langle\psi\varphi^*\rangle\right\}, \tag{71}$$

where

$$\langle\varepsilon\varphi^*\rangle = \langle\varepsilon\varphi\rangle^* = C\left[\omega^2\phi^* + 2\left(\frac{\delta\Gamma_2}{\delta\langle\varepsilon\psi^*\rangle}\right)^*\right]G'^* \tag{72}$$

$$\langle\psi\varphi^*\rangle = -\langle\psi^*\varphi\rangle^* = -\mathrm{i}G'^*. \tag{73}$$

The reason for equation (73) is that to obtain $\langle\psi\varphi^*\rangle$ from $\langle\psi^*\varphi\rangle$ we need both to take a complex conjugate and change $\psi \to -\psi$ in the path integral.

3.2. Charge conservation

We want to check if the conservation law in equation (16) holds in the mean on shell. Observe that

$$\mathrm{i}\langle J^\mu\rangle(x) = \phi^*(x)\nabla^\mu\phi(x) - \phi(x)\nabla^\mu\phi^*(x) + \left[\nabla_x^\mu - \nabla_{x'}^\mu\right]\langle\varphi(x)\varphi^*(x')\rangle\Big|_{x'=x}. \tag{74}$$

Therefore, from equation (35),

$$\mathrm{i}\nabla_\mu\langle J^\mu\rangle(x) + \phi^*j - \phi j^* = -\omega^2\phi^*\langle\varepsilon\varphi\rangle + \omega^2\phi\langle\varepsilon\varphi^*\rangle$$
$$+ \left[\mathbf{D}_0\langle\varphi(x)\varphi^*(x')\rangle - \langle\varphi(x)\varphi^*(x')\rangle\mathbf{D}_0\right]_{x'=x}. \tag{75}$$

Now, from the equations for the propagators, we have

$$\omega^2\phi(x)\langle\varepsilon(x)\varphi^*(x')\rangle + \mathbf{D}_0\langle\varphi(x)\varphi^*(x')\rangle = -2\int \mathrm{d}y\frac{\delta\Gamma_2}{\delta\langle\psi^*(x)X^s(y)\rangle}\langle X^s(y)\varphi^*(x')\rangle,$$

$$\omega^2\langle\varphi(x)\varepsilon(x')\rangle\phi^*(x') + \langle\varphi(x)\varphi^*(x')\rangle\mathbf{D}_0 = -2\int \mathrm{d}y\langle\varphi(x)X^s(y)\rangle\frac{\delta\Gamma_2}{\delta\langle X^s(y)\psi(x')\rangle}. \tag{76}$$

So, conservation in the mean holds if

$$\int \mathrm{d}y\left\{\frac{\delta\Gamma_2}{\delta\langle\psi(x)X^s(y)\rangle}\langle\varphi(x)X^s(y)\rangle - \frac{\delta\Gamma_2}{\delta\langle\psi^*(x)X^s(y)\rangle}\langle\varphi^*(x)X^s(y)\rangle\right\} = 0. \tag{77}$$

Equation (77) holds not only on shell, but identically. As we have seen, the 2PI EA is built from the expectation values of powers of the expression

$$\omega^2\int \mathrm{d}x\ \varepsilon\Big(\psi^*\varphi + \psi\varphi^*\Big), \tag{78}$$

which is obviously invariant under a transformation whereby

$$\delta\psi = \mathrm{i}\theta\varphi$$

$$\delta\psi^* = -\mathrm{i}\theta\varphi^*. \tag{79}$$

Equation (77) is just the Zinn–Justin identity associated with this symmetry [68, 74], and, therefore, conservation holds.

3.3. Intensity operator

To investigate the structure of the 2PI EA it is crucial to observe that the Feynman graphs which actually contribute to it have a rather peculiar structure. Namely, any graph containing 'matter' fields (φ, φ^*, ψ and ψ^*) comes from taking the expectation value of some term in the power expansion of

$$\left\langle \mathrm{e}^{\mathrm{i}\omega^2\int \mathrm{d}x\varepsilon\left(\psi^*\varphi+\psi\varphi^*\right)}\right\rangle. \tag{80}$$

Let us look into any such terms, containing N_φ φ fields, and similarly for the ψ, φ^* and ψ^* fields. It is obvious that $N_\varphi = N_{\psi^*}$ and $N_{\varphi^*} = N_\psi$.

Now, look at any Feynman graph obtained from the expectation value of that term. The graph will contain $L_{\langle\varphi\varphi\rangle}$ $\langle\varphi\varphi\rangle$ lines, and so on for all other (bosonic) propagators. The graph will be globally charge neutral, and also

$$N_\varphi = 2L_{\langle\varphi\varphi\rangle} + L_{\langle\varphi\varphi^*\rangle} + L_{\langle\varphi\psi\rangle} + L_{\langle\varphi\psi^*\rangle} + L_{\langle\varphi\varepsilon\rangle} \tag{81}$$

a similar analysis yields

$$N_{\psi^*} = L_{\langle\psi^*\varphi\rangle} + L_{\langle\psi^*\varphi^*\rangle} + L_{\langle\psi^*\psi\rangle} + 2L_{\langle\psi^*\psi^*\rangle} + L_{\langle\psi^*\varepsilon\rangle} \tag{82}$$

and so $N_\varphi = N_{\psi^*}$ implies

$$2L_{\langle\varphi\varphi\rangle} + L_{\langle\varphi\varphi^*\rangle} + L_{\langle\varphi\psi\rangle} + L_{\langle\varphi\varepsilon\rangle} = L_{\langle\psi^*\varphi^*\rangle} + L_{\langle\psi^*\psi\rangle} + 2L_{\langle\psi^*\psi^*\rangle} + L_{\langle\psi^*\varepsilon\rangle}. \tag{83}$$

Similarly,

$$2L_{\langle\varphi^*\varphi^*\rangle} + L_{\langle\varphi\varphi^*\rangle} + L_{\langle\varphi^*\psi^*\rangle} + L_{\langle\varphi^*\varepsilon\rangle} = L_{\langle\psi\varphi\rangle} + L_{\langle\psi\psi^*\rangle} + 2L_{\langle\psi\psi\rangle} + L_{\langle\psi\varepsilon\rangle}. \tag{84}$$

To find $G = \mathrm{i}\langle\varphi\psi^*\rangle$ and G^*, we must solve equation (65) and its conjugate. Now any graph contributing to the variational derivative must have

$$L_{\langle\psi\psi\rangle} = L_{\langle\psi\psi^*\rangle} = L_{\langle\psi^*\psi^*\rangle} = L_{\langle\psi^*\varepsilon\rangle} = L_{\langle\psi\varepsilon\rangle} = L_{\langle\psi^*\varphi^*\rangle} = L_{\langle\psi\varphi\rangle} = 0 \tag{85}$$

because the presence of any of those propagators would kill the graph on shell. It follows that it also must have

$$L_{\langle\varphi\varphi\rangle} = L_{\langle\varphi\varphi^*\rangle} = L_{\langle\varphi\psi\rangle} = L_{\langle\varphi\varepsilon\rangle} = L_{\langle\varphi^*\varphi^*\rangle} = L_{\langle\varphi\varphi^*\rangle} = L_{\langle\varphi^*\psi^*\rangle} = L_{\langle\varphi^*\varepsilon\rangle} = 0. \tag{86}$$

The only propagators left to build the graph with are G, G^* and C, and so we obtain a closed dynamics.

Observe that because of charge conservation at every vertex, matter fields (defined above) can only appear in closed matter loops, connected by C lines to themselves, other matter loops or ghost loops. Actually, graphs containing more than one matter loop cancel out in the derivative against the contribution of the graphs where each matter loop is replaced in turn by a ghost loop (a ghost loop having the same amplitude that a matter loop has, except for the inverse sign), so that only graphs with a single matter loop need to be considered. We shall discuss graphs containing C loops below.

Let us turn to $\langle\varphi(x)\varepsilon(x')\rangle$ and $\langle\varphi^*(x)\varepsilon(x')\rangle$. Now, we must solve equation (68), so we must seek graphs with $L_{\langle\varepsilon\psi^*\rangle} = 1$. Therefore, we shall have

$$2L_{\langle\varphi\varphi\rangle} + L_{\langle\varphi\varphi^*\rangle} + L_{\langle\varphi\psi\rangle} + L_{\langle\varphi\varepsilon\rangle} = 1. \tag{87}$$

Now, because $\langle\varepsilon\psi^*\rangle$ has charge -1, the remainder of the graph containing it must have charge $+1$. Therefore, we must have

$$L_{\langle\varphi\varepsilon\rangle} = 1 \tag{88}$$

and all other zero

$$L_{\langle\varphi\varphi\rangle} = L_{\langle\varphi\varphi^*\rangle} = L_{\langle\varphi\psi\rangle} = 0. \tag{89}$$

It follows that again we get a closed equation for $\langle\varepsilon\varphi\rangle$, and moreover this equation is linear.

A similar analysis shows that we get a linear, self-consistent equation for $\langle\varphi\varphi^*\rangle$. The equation to be solved is (71). Since we have already analyzed graphs with $L_{\langle\varepsilon\psi^*\rangle} = 1$, let us focus on the last term. We seek Feynman graphs with $L_{\langle\psi^*\psi\rangle} = 1$, since any such propagator surviving the derivative will kill the graph. This leaves two options, either $L_{\langle\varphi\varphi^*\rangle} = 1$ and $L_{\langle\varphi\varepsilon\rangle} = L_{\langle\varphi^*\varepsilon\rangle} = 0$, or else $L_{\langle\varphi\varphi^*\rangle} = 0$ and $L_{\langle\varphi\varepsilon\rangle} = L_{\langle\varphi^*\varepsilon\rangle} = 1$. Therefore, the third term can be written as the sum of two terms, one containing only C, G, G^*, $\langle\varepsilon\varphi\rangle$ and $\langle\varepsilon\varphi^*\rangle$, and the other linear in $\langle\varphi\varphi^*\rangle$. Observe, furthermore, that $\langle\varphi^*\psi\rangle = -\mathrm{i}G^*$ [recall equation (73)]. If we then compare equation (71) with equation (18), we conclude that we can identify the intensity operator with the operator in the term in equation (71), which is linear in $\langle\varphi\varphi^*\rangle$, namely,

$$I[z, y; z', y'] = -2\mathrm{i}\left\{\frac{\delta^2\Gamma_2}{\delta\langle\psi^*(z)\psi(z')\rangle\delta\langle\varphi(y)\varphi^*(y')\rangle} + \frac{\delta^2\Gamma_2}{\delta\langle\psi^*(z)\psi(z')\rangle\delta\langle\varphi^*(y')\varphi(y)\rangle}\right\}. \tag{90}$$

The two derivatives are necessary because in building the 2PI EA, $\langle\varphi\varphi^*\rangle$ and $\langle\varphi^*\varphi\rangle$ must be regarded as independent variables, although they will be equal to each other 'on shell', namely, when evaluated on a physical solution.

4. Loop expansion of the 2PI EA

In this section, we shall discuss in some detail the first three orders ($L = 1$, 2 and 3) in the loop expansion of the 2PI EA. To this effect, it is convenient to introduce a single 'matter' field multiplet $\chi^a = \left(\varphi, \varphi^*, \psi, \psi^*\right)$ and a 'ghost' field multiplet $\gamma^a = (\eta, \vartheta, \zeta, \xi)$. In terms of these fields, the quantum part of the EA Γ_2 can be written as

$$e^{i\Gamma_2} = \langle e^{\frac{i}{2}\omega^2 \int dx \varepsilon \sigma_{ab}\left(\chi^a\chi^b + i\gamma^a\gamma^b\right) - \sum_{n=3}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n f_n(x_1,\ldots,x_n)\varepsilon(x_1)\ldots\varepsilon(x_n)} \rangle_{2PI}, \quad (91)$$

where

$$\sigma_{ab} = \delta_a^1\delta_b^4 + \delta_a^2\delta_b^3 + \delta_a^3\delta_b^2 + \delta_a^4\delta_b^1. \quad (92)$$

Variations of the EA are derived from the rule

$$\frac{\delta\left\langle \chi^a(z_1)\chi^b(z_2)\right\rangle}{\delta\left\langle \chi^c(x_1)\chi^d(x_2)\right\rangle} = \delta_c^a\delta(z_1 - x_1)\delta_d^b\delta(z_2 - x_2). \quad (93)$$

The 2PI subscript in equation (91) means that to compute the 2PI EA, the exponential is expanded into its power series, then each expectation value is computed according to Wick's theorem, using full propagators in the internal lines, and finally non-2PI graphs are discarded.

The first observation is that the graphs so built have no external lines. Let us introduce the numbers V_n of n-point vertices, I of lines and L of loops.

$$\sum nV_n = 2I,$$
$$I - V = L - 1. \quad (94)$$

Eliminating I between these equations, we obtain

$$\sum(n-2)V_n = 2(L-1). \quad (95)$$

Therefore, if we consider graphs up to a certain number L of loops, only vertices with $n \leqslant 2L$ need to be considered, and only those with $n \leqslant L + 1$ will appear more than once in the same graph [73]. This fact makes the loop expansion the simplest perturbative approach to the evaluation of the 2PI EA.

The lowest (one loop) approximation corresponds to $L = 1$. Therefore, $V_n = 0$ for all n and $\Gamma_2^{(1)} = 0$. From equation (66), $\Sigma = Q = 0$. This is just the 'classical' approximation, where only coherent fields are considered.

4.1. Two loops theory

The following approximation is the two-loops one, where $L = 2$. This means $V_3 = 2$ and $V_n = 0$ for $n > 3$, or else $V_3 = 0$, $V_4 = 1$ and $V_n = 0$ for $n > 4$. Therefore, we can write

$$\begin{aligned}\Gamma_2^{(2)} = \frac{-i}{2}\Bigg\langle \Bigg\{ &\frac{i}{2}\omega^2 \int dx \varepsilon\sigma_{ab}\left(\chi^a\chi^b + i\gamma^a\gamma^b\right) \\ &- \frac{1}{6}\int dx_1dx_2dx_3\, f_3(x_1, x_2, x_3)\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\Bigg\}^2\Bigg\rangle_{2PI} \\ &+ \frac{i}{24}\left\langle \int dx_1dx_2dx_3dx_4\, f_4(x_1, x_2, x_3, x_4)\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\varepsilon(x_4)\right\rangle_{2PI}. \end{aligned} \quad (96)$$

To compute the self-energy and the intensity operator, we only require the Feynman graphs carrying matter fields in their internal lines. These are

$$\Gamma^{(2)}_{2,\mathrm{matter}} = \frac{\mathrm{i}}{4}\omega^4 \int \mathrm{d}z_1\mathrm{d}z_2\ C(z_1, z_2)\sigma_{a_1b_1}\sigma_{a_2b_2}\left\langle \chi^{a_1}(z_1)\chi^{a_2}(z_2)\right\rangle\left\langle \chi^{b_1}(z_1)\chi^{b_2}(z_2)\right\rangle. \quad (97)$$

We now derive the self-energy from equation (66) and the intensity operator from equation (90). Computing the variation of the EA with the rule equation (93), we immediately find the nonlinear and ladder approximations, respectively. We see that to this order there are no contributions from the non-Gaussianity in the ε fluctuations; these will show up at the next order.

4.2. Three loops theory

At three loops $L = 3$ the nontrivial possibilities are $V_3 = 4$, all others zero, $V_3 = 2$, $V_4 = 1$, all others zero, or else $V_4 = 2$, all others zero. Therefore,

$$\begin{aligned}
\Gamma^{(3)}_2 = \frac{-\mathrm{i}}{24}\Bigg\langle \Bigg\{ &\frac{\mathrm{i}}{2}\omega^2 \int \mathrm{d}x\varepsilon\sigma_{ab}\left(\chi^a\chi^b + \mathrm{i}\gamma^a\gamma^b\right) \\
&- \frac{1}{6}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ f_3(x_1, x_2, x_3)\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\Bigg\}^4\Bigg\rangle_{2\mathrm{PI}} \\
&+ \frac{\mathrm{i}}{2}\Bigg\langle\Bigg\{\frac{\mathrm{i}}{2}\omega^2 \int \mathrm{d}x\varepsilon\sigma_{ab}\left(\chi^a\chi^b + \mathrm{i}\gamma^a\gamma^b\right) \\
&- \frac{1}{6}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ f_3(x_1, x_2, x_3)\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\Bigg\}^2 \\
&\times \int \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}_3\mathrm{d}y_4\ f_4(y_1, y_2, y_3, y_4)\varepsilon(y_1)\varepsilon(y_2)\varepsilon(y_3)\varepsilon(y_4)\Bigg\rangle_{2\mathrm{PI}} \\
&- \frac{\mathrm{i}}{2}\Bigg\langle\Bigg\{\int \mathrm{d}x_1\mathrm{d}_2\mathrm{d}x_3\mathrm{d}x_4\ f_4(x_1, x_2, x_3, x_4)\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\varepsilon(x_4)\Bigg\}^2\Bigg\rangle_{2\mathrm{PI}}
\end{aligned} \quad (98)$$

The terms that contribute to the self-energy and the intensity operator can be written as

$$\begin{aligned}
\Gamma^{(3)}_{2,\mathrm{matter}} &= A + B \\
A &= \frac{-\mathrm{i}}{24}\frac{\omega^8}{16}\left\langle\left[\int \mathrm{d}x\varepsilon\sigma_{ab}\chi^a\chi^b\right]^4\right\rangle_{2\mathrm{PI}}, \\
B = \frac{1}{24}\frac{\omega^6}{12}&\left\langle\left[\int \mathrm{d}x\varepsilon\sigma_{ab}\chi^a\chi^b\right]^3 \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ f_3(x_1, x_2, x_3)\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\right\rangle_{2\mathrm{PI}}.
\end{aligned} \quad (99)$$

Explicitly,

$$\begin{aligned}
A = \frac{-\mathrm{i}\omega^8}{8}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\ &\sigma_{a_1b_1}\sigma_{a_2b_2}\sigma_{a_3b_3}\sigma_{a_4b_4}C(x_1, x_4)C(x_2, x_3) \\
&\times \left\langle \chi^{a_1}(x_1)\chi^{a_2}(x_2)\right\rangle\left\langle \chi^{b_1}(x_1)\chi^{a_3}(x_3)\right\rangle \\
&\times \left\langle \chi^{b_2}(x_2)\chi^{a_4}(x_4)\right\rangle\left\langle \chi^{b_3}(x_3)\chi^{b_4}(x_4)\right\rangle,
\end{aligned}$$

$$B = \frac{-\omega^6}{6} \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \ \sigma_{a_1 b_1} \sigma_{a_2 b_2} \sigma_{a_3 b_3} K(x_1, x_2, x_3)$$
$$\times \left\langle \chi^{a_1}(x_1) \chi^{a_2}(x_2) \right\rangle \left\langle \chi^{b_1}(x_1) \chi^{a_3}(x_3) \right\rangle \left\langle \chi^{b_2}(x_2) \chi^{b_3}(x_3) \right\rangle,$$
$$K(x_1, x_2, x_3) = -\int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \, f_3(y_1, y_2, y_3) C(x_1, y_1) C(x_2, y_2) C(x_3, y_3). \tag{100}$$

To compute the K kernel, we observe that

$$\frac{\delta \Gamma_2}{\delta f_3(x_1, x_2, x_3)} = \frac{\mathrm{i}}{6} \left\langle \varepsilon(x_1) \varepsilon(x_2) \varepsilon(x_3) \right\rangle_{2\mathrm{PI}}. \tag{101}$$

To leading order, we can substitute the two-loops expression, equation (96), for the EA, whereby we find

$$K(x_1, x_2, x_3) = \left\langle \varepsilon(x_1) \varepsilon(x_2) \varepsilon(x_3) \right\rangle_{2\mathrm{PI}}. \tag{102}$$

Therefore, K would vanish in a Gaussian theory. The self-energy and intensity operator have a similar structure

$$\Sigma(x_1, x_4) = \Sigma_A(x_1, x_4) + \Sigma_B(x_1, x_4), \tag{103}$$

$$\Sigma_A(x_1, x_4) = \omega^8 \int \mathrm{d}x_2 \mathrm{d}x_3 \ C(x_1, x_3) C(x_2, x_4) G(x_1, x_2) G(x_2, x_3) G(x_3, x_4), \tag{104}$$

$$\Sigma_B(x_1, x_4) = \omega^6 \int \mathrm{d}x_2 \ K(x_1, x_2, x_4) G(x_1, x_2) G(x_2, x_4), \tag{105}$$

$$\begin{aligned}
I[z, y; z', y'] &= I_A[z, y; z', y'] + I_B[z, y; z', y'], \\
I_A[z, y; z', y'] &= I_A^{(1)}[z, y; z', y'] + I_A^{(2)}[z, y; z', y'] + I_A^{(3)}[z, y; z', y'], \\
I_B[z, y; z', y'] &= I_B^{(1)}[z, y; z', y'] + I_B^{(2)}[z, y; z', y'], \\
I_A^{(1)}[z, y; z', y'] &= \omega^8 \delta(z - y) C(z', y') \int \mathrm{d}x \ C(z, x) G^*(z', x) G^*(x, y'), \\
I_A^{(2)}[z, y; z', y'] &= \omega^8 \delta(z' - y') C(z, y) \int \mathrm{d}x \ C(x, z') G(z, x) G(x, y), \\
I_A^{(3)}[z, y; z', y'] &= \omega^8 C(z, y') C(z', y) G(z, y) G^*(z', y'), \\
I_B^{(1)}[z, y; z', y'] &= \omega^6 \delta(z - y) K(z, z', y') G^*(z', y'), \\
I_B^{(2)}[z, y; z', y'] &= \omega^6 \delta(z' - y') K(z, z', y) G(z, y).
\end{aligned} \tag{106}$$

5. Application: overlapping intrusions on a homogeneous background

As an application, we will consider the case of a homogeneous medium with $\epsilon = \epsilon_0$, on which there are introduced spherical, overlapping bubbles with $\epsilon = \epsilon_1 = \epsilon_0 + \Delta\epsilon$. The bubbles all have the same radius R, and their centers are chosen independently at random with a homogeneous distribution over some control volume V. Let there be N bubbles centered at points $(\mathbf{x}_1, \ldots \mathbf{x}_N)$. Then,

$$\epsilon(\mathbf{x}) = \epsilon_0 - \Delta\epsilon \sum_{k=1}^{N} (-1)^k \sum_{i_1 < i_2 < \ldots < i_k} f_{i_1}(\mathbf{x}) \ldots f_{i_k}(\mathbf{x}), \tag{107}$$

where

$$f_i(\mathbf{x}) = \theta\left(1 - \frac{(\mathbf{x} - \mathbf{x}_i)^2}{R^2}\right) \equiv f(\mathbf{x} - \mathbf{x}_i). \tag{108}$$

Given this representation, we easily find

$$\bar{\epsilon} = \epsilon_0 + \Delta\epsilon\left(1 - \left(1 - \frac{v}{V}\right)^N\right), \tag{109}$$

where $v = 4\pi R^3/3$ is the volume of one bubble. We define

$$\varepsilon(\mathbf{x}) = \epsilon(\mathbf{x}) - \bar{\epsilon} = \epsilon(\mathbf{x}) - \Delta\epsilon + \Delta\epsilon\left(1 - \frac{v}{V}\right)^N. \tag{110}$$

Then,

$$C(\mathbf{x}, \mathbf{x}') = -\varepsilon_0^2 + \Delta\epsilon^2\left(1 + \frac{v}{V}(\gamma(\mathbf{x}, \mathbf{x}') - 2)\right)^N, \tag{111}$$

where

$$\varepsilon_0 = -\Delta\epsilon\left(1 - \frac{v}{V}\right)^N \tag{112}$$

and we have defined

$$\left\langle f_1(\mathbf{x}) f_1(\mathbf{x}')\right\rangle = \frac{v}{V}\gamma(\mathbf{x}, \mathbf{x}'), \tag{113}$$

$v\gamma$ is the volume of the intersection between a sphere centered at $\mathbf{x}$ and another centered at $\mathbf{x}'$. It is clear that $\gamma(0) = 1$ and $\gamma(\mathbf{x}) = 0$ if $\left(\mathbf{x} - \mathbf{x}'^2\right) \geqslant 4R^2$.

For the three-point correlations equation (102), we get

$$\begin{aligned} K(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = &-\Delta\epsilon^3\left(1 - \frac{v}{V}(\lambda(\mathbf{x}, \mathbf{x}', \mathbf{x}'') - \gamma(\mathbf{x}, \mathbf{x}') - \gamma(\mathbf{x}, \mathbf{x}'') - \gamma(\mathbf{x}', \mathbf{x}'') + 3)\right)^N \\ &-\varepsilon_0[C(\mathbf{x}, \mathbf{x}') + C(\mathbf{x}, \mathbf{x}'') + C(\mathbf{x}', \mathbf{x}'')] - \varepsilon_0^3, \end{aligned} \tag{114}$$

where we defined

$$\left\langle f_1(\mathbf{x}) f_1(\mathbf{x}') f_1(\mathbf{x}'')\right\rangle = \frac{v}{V}\lambda(\mathbf{x}, \mathbf{x}', \mathbf{x}''), \tag{115}$$

$v\lambda$ is the volume of the intersection of three spheres. The fact that K is nonvanishing shows that ε fluctuations are not Gaussian.

We are interested in the limit where $N, V \to \infty$, $N/V = \rho =$ constant. In this limit,

$$\bar{\epsilon} = \epsilon_0 + \Delta\epsilon\left(1 - \mathrm{e}^{-\rho v}\right), \tag{116}$$

$$C(\mathbf{x}, \mathbf{x}') = \Delta\epsilon^2 \mathrm{e}^{-2\rho v}\left[\mathrm{e}^{\rho v\gamma(\mathbf{x}, \mathbf{x}')} - 1\right]. \tag{117}$$

C vanishes when the distance between its arguments exceeds $2R$, and

$$K(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = -\Delta\epsilon^3 \mathrm{e}^{-3\rho v} \Big\{ \mathrm{e}^{-\rho v \left(\lambda\left(\mathbf{x},\mathbf{x}',\mathbf{x}''\right) - \gamma\left(\mathbf{x},\mathbf{x}'\right) - \gamma\left(\mathbf{x},\mathbf{x}'\right) - \gamma\left(\mathbf{x}'',\mathbf{x}''\right)\right)} - \left[\mathrm{e}^{\rho v \gamma\left(\mathbf{x},\mathbf{x}'\right)} + \mathrm{e}^{\rho v \gamma\left(\mathbf{x},\mathbf{x}''\right)} + \mathrm{e}^{\rho v \gamma\left(\mathbf{x}',\mathbf{x}''\right)} \right] + 2 \Big\}. \tag{118}$$

K vanishes when the distance between any two arguments exceeds $2R$. ρv is the fraction of the total volume occupied by the spheres, not counting overlapping. If they are sparse, then $\rho v \ll 1$, In this limit, the formulas above simplify further to

$$\bar{\epsilon} = \epsilon_0 + \Delta\epsilon\rho v, \tag{119}$$

$$C(\mathbf{x}, \mathbf{x}') = \Delta\epsilon^2 \rho v \gamma(\mathbf{x}, \mathbf{x}'), \tag{120}$$

$$K(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = \Delta\epsilon^3 \rho v \lambda(\mathbf{x}, \mathbf{x}', \mathbf{x}''). \tag{121}$$

5.1. Non-Gaussian contributions to the diffuse intensity

We shall conclude by analyzing in some detail the non-Gaussian contributions to the diffuse intensity. This is measured by the two-point correlation

$$G_1(\mathbf{x}, \mathbf{x}') = \langle \varphi(\mathbf{x}) \varphi^*(\mathbf{x}') \rangle. \tag{122}$$

If we assume the mean field is just a plane wave, $\phi = (2\pi)^{-3/2} \phi_0 \mathrm{e}^{\mathrm{i}\mathbf{K}\mathbf{x}}$, then the problem is translation-invariant and we can Fourier transform all relevant quantities, e.g.

$$G_1(x, x') = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \mathrm{e}^{\mathrm{i}\mathbf{p}\left(\mathbf{x}-\mathbf{x}'\right)} G_1(\mathbf{p}). \tag{123}$$

We similarly introduce the Fourier transforms $G(\mathbf{p})$ and $Q(\mathbf{p})$ for the propagator and the self-energy, respectively. The intensity operator becomes

$$I[\mathbf{z}, \mathbf{y}; \mathbf{z}', \mathbf{y}'] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 p'}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \, \mathrm{e}^{\mathrm{i}\mathbf{p}(\mathbf{z}-\mathbf{y})} \mathrm{e}^{-\mathrm{i}\mathbf{p}'\left(\mathbf{z}'-\mathbf{y}'\right)} \mathrm{e}^{\mathrm{i}\mathbf{q}\left(\mathbf{y}-\mathbf{y}'\right)} I[\mathbf{p}, \mathbf{p}'; \mathbf{q}]. \tag{124}$$

The Bethe–Salpeter equation reads

$$G_1(\mathbf{p}) = |G(\mathbf{p})|^2 \left\{ \phi_0^2 \mathcal{I}[\mathbf{p}, \mathbf{K}] + \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \mathcal{I}[\mathbf{p}, \mathbf{q}] G_1(\mathbf{q}) \right\}, \tag{125}$$

where

$$\mathcal{I}[\mathbf{p}, \mathbf{K}] = I[\mathbf{p}, \mathbf{p}; \mathbf{p} - \mathbf{K}]. \tag{126}$$

Since flux is conserved, for there to be scattered intensity, the mean field equation must display absorption, namely, $Q(\mathbf{p})$ must be complex. Indeed, the Ward identity reads

$$\mathrm{Im} Q(\mathbf{q}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \mathrm{Im} G(\mathbf{p}) \mathcal{I}[\mathbf{p}, \mathbf{q}], \tag{127}$$

where, moreover, since $G(\mathbf{p}) = \left[p^2 - \bar{\epsilon}\omega^2 - Q(\mathbf{p}) \right]^{-1}$,

$$\mathrm{Im} G(\mathbf{p}) = |G(\mathbf{p})|^2 \, \mathrm{Im} Q(\mathbf{p}). \tag{128}$$

Each term in the perturbative expansion of I [equations (21) and (106)] yields a corresponding term in the expansion of $\mathcal{I}$. Let us expand on the structure of some of these terms.

The leading Gaussian contribution to the scattered intensity is given by the ladder approximation

$$I[\mathbf{p},\, \mathbf{p}';\, \mathbf{q}] = \omega^4 C(\mathbf{q}). \tag{129}$$

Observe that

$$v\gamma(\mathbf{x},\, \mathbf{x}') = (2\pi)^3 \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\, e^{i(\mathbf{px}+\mathbf{p}'\mathbf{x}')} f(\mathbf{p}) f(\mathbf{p}')\delta(\mathbf{p}+\mathbf{p}'), \tag{130}$$

where

$$f(\mathbf{p}) = \frac{3v}{(Rp)^3}[\sin(Rp) - Rp\cos(Rp)]. \tag{131}$$

$f(\mathbf{p})$ is essentially flat (and equal to v) up to wave numbers of order $1/R$. Then, the approximation in equation (120) gives

$$I_{\text{ladder}}[\mathbf{p},\, \mathbf{p}';\, \mathbf{q}] = \omega^4 \Delta\epsilon^2 \rho f(\mathbf{q})^2. \tag{132}$$

The non-Gaussian contributions to the intensity operator are given by the last two lines in equation (106). Write

$$v\lambda(\mathbf{x},\, \mathbf{x}',\, \mathbf{x}'') = (2\pi)^3 \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\, e^{i(\mathbf{px}+\mathbf{p}'\mathbf{x}'+\mathbf{p}''\mathbf{x}'')}\, \delta\left(\mathbf{p}+\mathbf{p}'+\mathbf{p}''\right) f(\mathbf{p}) f\left(\mathbf{p}'\right) f\left(\mathbf{p}''\right) \tag{133}$$

then,

$$I_B[\mathbf{p},\, \mathbf{p}';\, \mathbf{q}] = \omega^6 \Delta\epsilon^3 \rho f(\mathbf{q}) \int \frac{d^3r}{(2\pi)^3}\left\{ G^*(\mathbf{r}) f(\mathbf{r}-\mathbf{p}') f(\mathbf{p}'-\mathbf{q}-\mathbf{r}) \right. \\ \left. + G(\mathbf{r}) f(\mathbf{p}-\mathbf{r}) f(\mathbf{q}+\mathbf{r}-\mathbf{p}) \right\}. \tag{134}$$

After this analysis, it is straightforward to build the kernel $\mathcal{I}$. We give a diagrammatic representation in figures 1–8. We represent $\mathcal{I}[\mathbf{p},\, \mathbf{K}]$ as a four-terminal device, with momentum $\mathbf{p}$ flowing into the upper left vertex and being extracted from the lower left vertex, and momentum $\mathbf{K}$ flowing into the lower right vertex and being extracted from the upper right vertex; momentum conservation holds in internal vertices. We represent G by a right pointing arrow, G^* by a left pointing one, f by a wavy line and G_1 by a double arrow. Points indicate that no propagator is assigned to that line. With these conventions (see figure 1) the Bethe–Salpeter equation (125) looks like figure 2. The different contributions to $\mathcal{I}[\mathbf{p},\, \mathbf{K}]$ are shown in the remaining figures.

When $\mathbf{p},\, \mathbf{K} \ll R^{-1}$ both $\mathcal{I}_{\text{ladder}}$ and $\mathcal{I}_B$ converge to constant values, $\omega^4\Delta\epsilon^2\rho v^2$ and (modulo a numerical factor) $\omega^4\Delta\epsilon^2\rho v^2\left(\omega^2\Delta\epsilon R^2\right)$, respectively. The other two loops' contributions are suppressed with respect to $\mathcal{I}_{\text{ladder}}$ and $\mathcal{I}_B$ by a factor of ρv, which we assume is small. However, they remain nontrivial functions of the wave numbers, even at long wavelengths; the maximally crossed graph $\mathcal{I}_A^{(3)}$, in particular, is related to the backscattering peak, as can be appreciated from figure 8.

5.2. The self-energy

When estimating the corrections to the self-energy, we make the approximation of replacing the self-energy by an effective inverse square speed of sound, namely,

$$\epsilon_{\rm eff} = \bar{\epsilon} + \frac{1}{\omega^2}\int \mathrm{d}y\ \Sigma(x, y). \tag{135}$$

Under this approximation solving for G is immediate

$$G(r) = \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{pr}}}{p^2 - k^2}, \tag{136}$$

where $k = \omega\epsilon_{\rm eff}^{1/2}$. The variable k is generally complex, with both its real and imaginary parts being positive. We identify three contributions to the self-energy

$$\Sigma = \Sigma_G^1 + \Sigma_{NG}^2 + \Sigma_G^2. \tag{137}$$

The first one is the usual (one-loop), nonlinear approximation in equation (20) (see figure 9), the second is the two-loops non-Gaussian term in equation (105) (see figure 10), and the third is the two-loops Gaussian correction in equation (104) (see figure 11). We write, correspondingly,

$$\epsilon_{\rm eff} = \bar{\epsilon} + \epsilon_G^1 + \epsilon_G^2 + \epsilon_{NG}^2. \tag{138}$$

Using equations (20), (120), (130) and (139), we obtain

$$\epsilon_G^1 = \omega^2\Delta\epsilon^2\rho\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{f^2(p)}{p^2 - k^2}, \tag{139}$$

where $f(p)$ is given by equation (131). Explicit evaluation yields

$$\epsilon_G^1 = \frac{3}{2}\omega^2R^2\Delta\epsilon^2\rho v\ \mathcal{G}_1(kR), \tag{140}$$

where

$$\mathcal{G}_1(x) = \frac{\mathrm{i}}{x^5}\left[1 - \mathrm{e}^{2\mathrm{i}x} + x^2 + x^2\mathrm{e}^{2\mathrm{i}x} + 2\mathrm{i}x\mathrm{e}^{2\mathrm{i}x} + \frac{2\mathrm{i}x^3}{3}\right]. \tag{141}$$

When $kR \to 0$, this becomes

$$\mathcal{G}_1(kR) = \frac{4}{15} + \frac{2}{9}\mathrm{i}kR - \frac{4}{35}(kR)^2 - \frac{2}{45}\mathrm{i}(kR)^3 + \cdots. \tag{142}$$

Similarly, from equations (105), (121) and (133)

$$\epsilon_{NG}^2 = \frac{\omega^4\Delta\epsilon^3\rho}{(2\pi)^6}\int \mathrm{d}^3p\mathrm{d}^3q\frac{f(p)f(q)f(\mathbf{p}+\mathbf{q})}{\left(p^2 - k^2\right)\left(q^2 - k^2\right)}. \tag{143}$$

Explicit evaluation yields

$$\epsilon_{NG}^2 = \frac{3}{4}\omega^4R^4\Delta\epsilon^3\rho v\ \mathcal{G}_2(kR), \tag{144}$$

where

$$\mathcal{G}_2(x) = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_1(x) + \frac{\mathrm{i}}{2x^7}\left(\mathrm{e}^{2\mathrm{i}x}(1 - \mathrm{i}x) - 1 - \mathrm{i}x\right)^2. \tag{145}$$

When $kR \to 0$ this becomes

$$\mathcal{G}_2(kR) = \frac{68}{315} + \frac{2}{9}\mathrm{i}kR + \cdots. \tag{146}$$

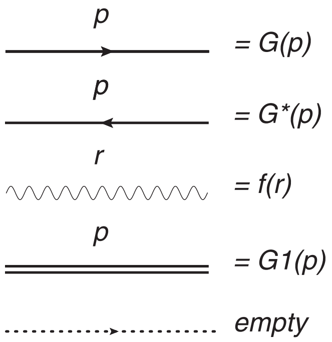

Figure 1. The graphic code we shall use in this paper. We represent G by a right pointing arrow, G^* by a left pointing one, f by a wavy line and G_1 by a double arrow. Points indicate that no propagator is assigned to that line.

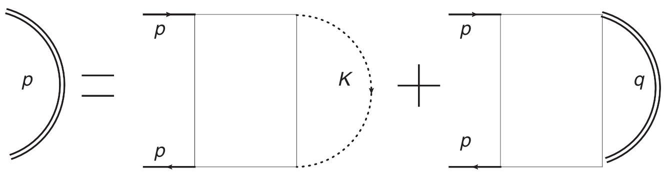

Figure 2. The Bethe–Salpeter equation (125) in graphic representation. A factor of ϕ_0^2 is implicit in the first term in the right-hand side.

Finally, from equation (104)

$$\epsilon_G^2 = \omega^6 \Delta\epsilon^4 \rho^2 \frac{1}{(2\pi)^6} \int \mathrm{d}^3p\mathrm{d}^3q \frac{f^2(p) f^2(q)}{\left(p^2 - k^2\right)\left(q^2 - k^2\right)\left((\mathbf{p} + \mathbf{q})^2 - k^2\right)}. \quad (147)$$

The integral is not analytic in the limit $kR \to 0$, the leading term being

$$\epsilon_G^2 \approx \frac{1}{9} \omega^6 R^6 \Delta\epsilon^4 (\rho v)^2 \ln(kR). \quad (148)$$

Already from the parametric dependence in these estimates we see that when $\omega^2 R^2 \Delta\epsilon^2 \rho v \leqslant \bar{\epsilon}$, corrections to $\bar{\epsilon}$ will be small. If $\omega^2 R^2 \Delta\epsilon \leqslant 1$, then $\epsilon_G^1 \geqslant \epsilon_{NG}^2$, and if, moreover, $\omega^2 R^2 \Delta\epsilon \rho v \ln\left(\bar{\epsilon}^{1/2} \omega R\right) \leqslant 1$, then $\epsilon_{NG}^2 \geqslant \epsilon_G^2$ also. Therefore, if two-loops Gaussian terms are included, e.g., to restore reciprocity without spoiling flux conservation, it would be simply inconsistent not to include also the non-Gaussian corrections. Observe that the effective speed of sound is complex, but the imaginary part is suppressed by a further power of ωR.

We can analyze the opposite situation where $\omega^2 R^2 \gg 1$. In this case, we expect kR will be large as well, so we can neglect oscillatory factors in ϵ_G^1 and ϵ_{NG}^2. Then, these corrections become $\epsilon_G^1 \approx (\omega/k)^2 \Delta\epsilon^2 \rho v$ and $\epsilon_{NG}^2 \approx (\omega/k)^4 \Delta\epsilon^3 \rho v$. We see that unless $\Delta\epsilon$ is unrealistically large, these corrections will not be large. As for ϵ_G^2, an inspection of equation (147) suggests that for large kR

$$\epsilon_G^2 = \left(\frac{\omega}{k}\right)^6 \Delta\epsilon^4 \rho^2 \left(\int \frac{\mathrm{d}^3p}{(2\pi)^3} f^2(p)\right)^2 \quad (149)$$

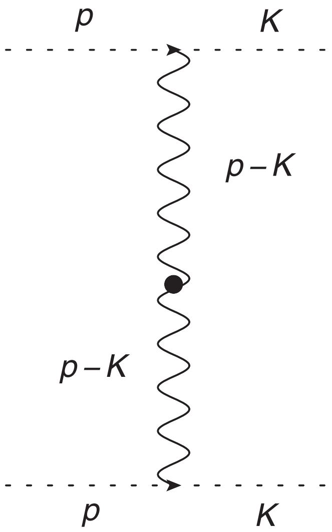

Figure 3. The term in $\mathcal{I}$ from the ladder approximation.

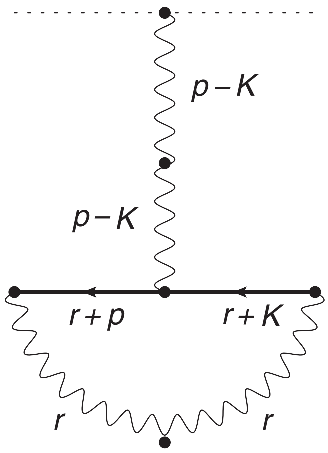

Figure 4. The term in $\mathcal{I}$ from $I_A^{(1)}$ in equation (106).

and

$$\int \frac{d^3p}{(2\pi)^3} f^2(p) = \int d^3r\, f^2(r) = \int d^3r\, f(r) = v. \tag{150}$$

So also this correction is not large, $\epsilon_G^2 \approx (\omega/k)^6 \Delta\epsilon^4 (\rho v)^2$. As long as $\Delta\epsilon$ is small, we obtain the same hierarchy as before.

6. Conclusions

In this paper, we have formulated the problem of scalar wave propagation in a random medium in a field theoretic language that connects it immediately to the larger body of work

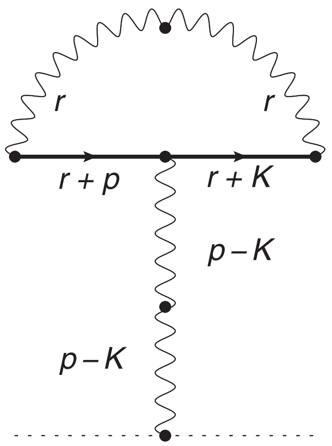

Figure 5. The term in $\mathcal{I}$ from $I_A^{(2)}$ in equation (106).

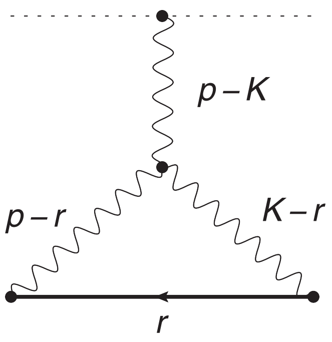

Figure 6. The non-Gaussian term in $\mathcal{I}$ from $I_B^{(1)}$ in equation (106).

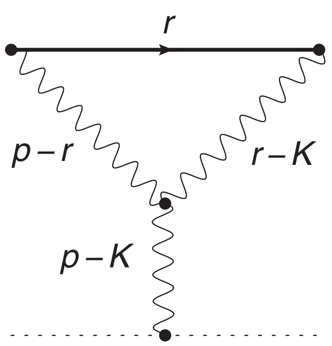

Figure 7. The non-Gaussian term in $\mathcal{I}$ from $I_B^{(2)}$ in equation (106).

[74, 75] addressed to similar problems in high-energy physics and cosmology [57] and the theory of turbulence [76–80]. The advantages of the method, compared with the straightforward approach of iterating the Dyson equation, are that it provides a partial resummation of

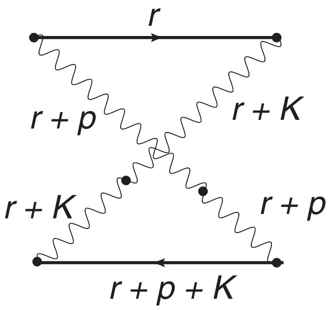

Figure 8. The term in $\mathcal{I}$ from the maximally crossed graph $I_A^{(3)}$ in equation (106). Observe that $\mathbf{K}$ and $\mathbf{p}$ appear as $\mathbf{K} + \mathbf{p}$; no other graph has this feature at this order in the loop expansion.

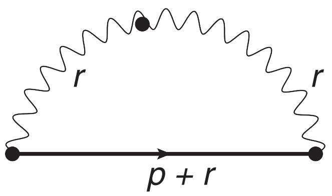

Figure 9. The one-loop contribution to the self-energy, which reproduces the so-called nonlinear approximation.

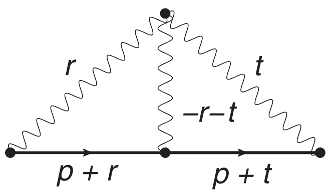

Figure 10. The two-loops contribution to the self-energy from non-Gaussian effects.

the perturbative series, which avoids overcounting and has energy conservation built-in order-by-order [69]. As a sample of the power of the method, we have analyzed the effect of non-Gaussian statistics on the mass and intensity operators. We have shown that in a generic model, the non-Gaussian effects are quantitatively dominant over Gaussian ones at the two-loops order. The Gaussian effects are nevertheless relevant to the understanding of such phenomena as coherent backscattering [10], and for this reason they cannot be simply discarded; this phenomenon is beyond the cases discussed in this paper.

In the last analysis, we believe the final proof of the usefulness of the language we are proposing will be given by its application to the electromagnetic case and the important problem of depolarization. There are two features of the field theory approach which we believe make further inquiry worthwhile.

First, in the problem of scattering of a wave at a random surface separating two media, usually two small parameters are identified, namely, the ratio of a typical height of the surface to wavelength, and the ratio of wavelength to the typical surface radius of curvature. Perturbation theory in the first parameter leads to the small perturbation method (SPM), while

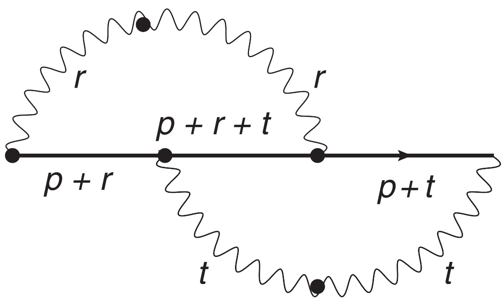

Figure 11. The two-loops contribution to the self-energy from Gaussian correlations.

perturbation in the second leads to the Kirchhoff approximation (KA); more sophisticated approaches combine features of both [4]. The field theoretic method, on the other hand, takes as a small parameter the contrast in the speed of sound in the media above and below the surface. For this reason, the field theoretic method does not assume surface protrusions are small, nor that the surface is smooth, and can be used to explore regions of parameter space where both the SPM and KA are beyond their range of applicability.

Another feature of the field theory language which could prove decisive is its flexibility towards the application of even more powerful nonperturbative techniques, among which renormalization group improvement of the self-energy and intensity operators feature prominently [81]. Renormalization group methods have been applied throughout theoretical physics; most relevant to the present discussion is the application to the propagation of elastic waves in disordered two-dimensional media [82, 83]. The application of the renormalization group method in the context of the 2PI EA is developed in [84–90]. We hope to report shortly on progress in these directions.

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