Comparison of classical and quantum mechanical uncertainties

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A comparison of classical and quantum uncertainties is presented for the particle-in-a-box, the harmonic oscillator, and the one-electron atom. It is found that the quantum results reduce to the classical in the limit of either very large quantum numbers or $h \to 0$. A classical uncertainty principle is derived and compared with its quantum analogue. A possible relationship between zero-point motion and the uncertainty principle is noted.

I. INTRODUCTION

The usual textbook development of the uncertainty principle[1-3] generally follows Heisenberg's original elaboration where the inequality

$$\Delta x \Delta p_x \geq h/4\pi, \tag{1}$$

is presented as a uniquely quantum-mechanical result which sets a lower bound on the product of the observational uncertainties in a single simultaneous measurement of a particle's position and momentum.[4] Although this approach is commended by its heuristic success, it is somewhat pedagogically incomplete because the initiate to quantum mechanics recognizes little similarity between this discussion and his previous experiences with classical physics. Additionally the statistical interpretation[5-8] of the uncertainty principle, which emphasizes that Eq. (1) applies not to observational uncertainties in a single measurement but instead to the position and momentum standard deviations in an ensemble of similarly prepared systems, is not always developed.[1]

Brown[9] has recently demonstrated, in the limit of $h \to 0$, that the modulus squared of the WKB approximation to the wave-packet solution of the Schrödinger equation approaches the classical probability density for a system whose initial coordinates have been imperfectly specified. Consequently a comparison of uncertainties calculated from the classical probability density with those obtained from quantum mechanics would appear to be of interest. We thus present herein such a comparison for the particle in a box, the harmonic oscillator, and the one-electron atom; additionally, we derive a classical uncertainty principle and compare it with its quantum-mechanical counterpart. Also, where appropriate, the above results are discussed in terms of the statistical interpretation of the uncertainty principle.

II. PRELIMINARIES

The uncertainty, both classically and quantum mechanically, of any quantity F is defined to be equal to the standard deviation about its mean, i.e.,

$$\Delta F = (\overline{F^2} - \overline{F}^2)^{1/2}, \tag{2}$$

where $\overline{F}$ represents the mean value of F.

Classically, the mean value of any quantity F over a time interval τ is calculated from[10]

$$\overline{F} = \int_0^\tau F(t)dt \Big/ \int_0^\tau dt, \tag{3}$$

or equivalently, where $v = dx/dt$ is the classical velocity,

$$\overline{F} = \int_{x_1}^{x_2} F(x)v(x)^{-1}dx \Big/ \int_{x_1}^{x_2} v(x)^{-1}dx. \tag{4}$$

In quantum mechanics $\overline{F}$ is calculated from a knowledge of the system's normalized wavefunction $\psi(x)$ through the expression

$$\overline{F} = \int_{x_1}^{x_2} \psi(x)^* F(x)\psi(x)dx. \tag{5}$$

III. PARTICLE IN A BOX

We consider a particle moving under the influence of the potential

$$V(x) = \infty, \ x \leq 0,$$
$$V(x) = 0, \ \text{for } 0 \leq x \leq l,$$
$$V(x) = \infty, \ x \geq l, \tag{6}$$

i.e., it moves freely along the interval $0 < x < l$. A determination of its position uncertainty requires a knowledge of $\overline{x}$ and $\overline{x^2}$ over one full period of its motion.

Since the magnitude of the particle's velocity is constant, the classical values of the latter quantities are conveniently calculated from Eq. (4). It is found that

$$\overline{x} = v^{-1}\left(\int_0^l xdx - \int_l^0 xdx\right) \times \left[v^{-1}\left(\int_0^l dx - \int_l^0 dx\right)\right]^{-1} = \frac{l}{2} \tag{7}$$

and

$$\overline{x^2} = v^{-1}\left(\int_0^l x^2dx - \int_l^0 x^2dx\right) \times \left[v^{-1}\left(\int_0^l dx - \int_l^0 dx\right)\right]^{-1} = \frac{l^2}{3}. \tag{8}$$

Thus the classical position uncertainty is given by

$$\Delta x_{cl} = (\overline{x^2} - \overline{x}^2)^{1/2} = l/2\sqrt{3}. \tag{9}$$

For a quantum mechanical particle in a box of length l, it is known[10] that

$$(\overline{x^2} - \overline{x}^2) = l^2(1 - 6/n^2\pi^2)/12, \tag{10}$$

where n, the quantum number of the nth stationary state, is a positive integer. Thus the corresponding position uncertainty is

$$\Delta x_q = l(1 - 6/n^2\pi^2)^{1/2}/2\sqrt{3}. \tag{11}$$

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Table I Quantum and classical position uncertainties for a particle in a 10-m box.

n	$\Delta x_q(m)$	$\Delta x_{cl}(m)$
1	1.8076	2.8868
5	2.8514	2.8868
10	2.8780	2.8868
100	2.8867	2.8868
1000	2.8868	2.8868

Upon comparing Eqs. (9) and (11), it is indeed found that Δx_q approaches Δx_{cl} in the classical limit of $n \to \infty$. It is also observed that both Δx_q and Δx_{cl} are independent of the particle's mass; consequently the position uncertainty of a garbage truck in a given box will be the same as that for an electron. Of greater note however, and in contrast to the usual expectation, Δx_q is found to be less than Δx_{cl}; this is illustrated in Table I for a particle in a 10-m box. Table I also suggests that Δx_q indeed represents an ensemble standard deviation and not an observational uncertainty in a single measurement. Namely if the latter interpretation is correct, one would find, depending upon the particle's energy, that its position in a 10-m box could not be determined more precisely than plus or minus several meters in a single measurement. However if it is the former interpretation which is valid, one would find that the standard deviation about the mean for an ensemble of position measurements would range, depending upon the particle's energy, from 1.8076 to 2.8868 m. Since we can generally determine the position of a macroscopic particle within at least several centimeters, one suspects, at first blush, that it is the ensemble interpretation that contains a greater measure of truth.

The momentum uncertainty is determined by calculating $\bar{p}$ and $\overline{p^2}$. Classically, the average momentum of a particle, such as one in a box, which is executing periodic motion is zero,[11] i.e.,

$$\bar{p} = 0; \tag{12}$$

since the energy of a particle in a box is given by

$$E = p^2/2m, \tag{13}$$

we find that

$$\overline{p^2} = 2mE, \tag{14}$$

and therefore the classical momentum uncertainty is

$$\Delta p_{cl} = (\overline{p^2} - \bar{p}^2)^{1/2} = \sqrt{2mE}\,. \tag{15}$$

The development of Δp_q parallels that of Δp_{cl}. That is, since the average momentum of a quantum-mechanical system described by a real wave function is zero, and since Eq. (14) also applies to the permitted quantum levels E_n, one can write

$$\Delta p_q = (\overline{p^2} - \bar{p}^2)^{1/2} = \sqrt{2mE_n}\,, \tag{16}$$

for the quantum-mechanical momentum uncertainty.

Obviously for the allowed quantum levels, $\Delta p_q = \Delta p_{cl}$ for the particle in a box. Additionally, it is observed that the physical origin of Δp_{cl} resides in the vector nature of momentum. That is, although from Eq. (13) the magnitude of the momentum of a particle of energy E will be known prior to observation, its direction will be unknown. Consequently in a very large number of observations on similarly prepared systems, half of the particles will be found to have momentum $p = -mv$ and half a momentum $p = mv$. Thus $\bar{p} = 0$, $\overline{p^2} = m^2v^2 = 2mE$, and $\Delta p_{cl} = \sqrt{2mE}$. It is tempting to apply a similar interpretation to the origin of Eq. (16).

IV. HARMONIC OSCILLATOR

A harmonically oscillating particle is subject to the potential

$$V(x) = kx^2/2, \tag{17}$$

where k is the force constant. In classical mechanics, the position of such a particle of mass m is expressed as[12]

$$x(t) = A \sin(\omega t + \phi_0), \tag{18}$$

where A is the amplitude of the motion, ω is the angular frequency given by

$$\omega = (k/m)^{1/2}, \tag{19}$$

and ϕ_0 is the initial phase. Consequently, $\bar{x}$ and $\overline{x^2}$ are best calculated by employing Eq. (3). For $\tau = 2\pi/\omega$, the period of oscillation, it is found that

$$\bar{x} = A \int_0^\tau \sin(\omega\tau + \phi_0)dt \Big/ \int_0^\tau dt = 0 \tag{20}$$

and

$$\overline{x^2} = A^2 \int_0^\tau \sin^2(\omega\tau + \phi_0)dt \Big/ \int_0^\tau dt = \frac{A^2}{2}. \tag{21}$$

Since the energy of the oscillator is $E = kA^2/2$, one can also write, with the aid of Eq. (19), that

$$\overline{x^2} = E/m\omega^2. \tag{22}$$

Thus the classical position uncertainty for the harmonic oscillator is

$$\Delta x_{cl} = (\overline{x^2} - \bar{x}^2)^{1/2} = (E/m\omega^2)^{1/2}. \tag{23}$$

The quantum-mechanical values of $\bar{x}$ and $\overline{x^2}$ for the harmonic oscillator[12] are, respectively,

$$\bar{x} = 0 \tag{24}$$

and $\overline{x^2} = E_n/m\omega^2$, where E_n is the energy of the nth quantum level. Consequently, the quantum-mechanical position uncertainty for the harmonic oscillator is

$$\Delta x_q = (\overline{x^2} - \bar{x}^2)^{1/2} = (E_n/m\omega^2)^{1/2}, \tag{26}$$

which, for $E = E_n$, is identical to the classical expression.

Equation (26) also suggests that Δx_q represents an ensemble standard deviation. Namely, for a 1-kg particle oscillating harmonically with 1 J of energy on the end of a spring with force constant 1 N/m, one calculates that $\Delta x_q = \pm 1$ m which is 70.7% of the classical amplitude of 1.414 m. Again, it would seem that the position of the particle could, in actuality, be determined much more precisely in any given single measurement.

Since $\bar{p} = 0$ for both the classical and quantum-mechanical cases, the determination of the momentum uncertainty simply requires knowledge of $\overline{p^2}$ which, from the virial theorem, is found classically to be given by

$$\overline{p^2} = mk\overline{x^2} = mE; \tag{27}$$

and quantum mechanically by

$$\overline{p^2} = mk\overline{x^2} = mE_n. \quad (28)$$

Thus, the classical and quantum-mechanical momentum uncertainties are found, respectively, to be

$$\Delta p_{cl} = (\overline{p^2} - \bar{p}^2)^{1/2} = (mE)^{1/2} \quad (29)$$

and

$$\Delta p_q = (\overline{p^2} - \bar{p}^2)^{1/2} = (mE_n)^{1/2}; \quad (30)$$

or, for $E = E_n$, $\Delta p_q = \Delta p_{cl}$.

Recalling the harmonically oscillating 1-kg mass discussed above, its quantum-mechanical momentum uncertainty is calculated to be ± 1 kg m/sec which corresponds to a velocity uncertainty of ± 1 m/sec. Thus it would here also seem relatively straightforward to test whether Eq. (30) applies to observational uncertainties of single measurements; considering that the magnitude of the classical velocity varies from 0 to 1.414 m/sec, one suspects that it does not.

V. ONE-ELECTRON ATOM

From classical mechanics,[13] the radial position of a particle moving in an elliptical trajectory under the influence of a central force $F = K/r^2$, with $K = -Ze^2$ for a one-electron atom, is known to be

$$r = a(1 - \epsilon^2)/(1 + \epsilon \cos\theta), \quad (31)$$

where r is measured from the center of the force to the particle; θ is the angle, measured in a counterclockwise sense, between the major axis of the ellipse and the radial vector; ϵ is the ellipse's eccentricity, i.e.,

$$\epsilon = (1 + 2EL^2/mK^2)^{1/2}, \quad (32)$$

with L representing the particle's angular momentum; and a is given by

$$a = |-K/2E|. \quad (33)$$

The mean value of r is calculated by first recalling[13] that

$$d\theta/dt = L/mr^2 \quad (34)$$

and then using Eqs. (3) and (31) to write

$$\bar{r} = \int_0^{2\pi} r^3(\theta)d\theta \Big/ \int_0^{2\pi} r^2(\theta)d\theta$$
$$= \text{a}(1 - \epsilon^2) \int_0^{2\pi} (1 + \cos\theta)^{-3}$$
$$\times d\theta \Big/ \int_0^{2\pi} (1 + \epsilon\cos\theta)^{-2}\, d\theta. \quad (35)$$

The integrals in Eq. (35) are executed in the Appendix and found to be

$$\int_0^{2\pi} (1 + \cos\theta)^{-2}\, d\theta = \frac{2\pi}{(1 - \epsilon^2)^{3/2}} \quad (\text{A7})$$

and

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-3} d\theta = \frac{(2 + \epsilon^2)\pi}{(1 - \epsilon^2)^{5/2}}; \quad (\text{A10})$$

thus

$$\bar{r} = a(1 + \epsilon^2/2). \quad (36)$$

In a similar manner, $\overline{r^2}$ is written

$$\overline{r^2} = \int_0^{2\pi} r^4(\theta)d\theta \Big/ \int_0^{2\pi} r^2(\theta)d\theta$$
$$= a^2(1 - \epsilon^2) \int_0^{2\pi} (1 + \epsilon\cos\theta)^{-4}$$
$$\times d\theta \Big/ \int_0^{2\pi} (1 + \epsilon\cos\theta)^{-2} d\theta, \quad (37)$$

with

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-4} d\theta = \frac{(2 + 3\epsilon^2)\pi}{(1 - \epsilon^2)^{7/2}}; \quad (\text{A13})$$

therefore

$$\overline{r^2} = a^2(3\epsilon^2 + 2)/2. \quad (38)$$

The classical uncertainty in the radial position of the particle is then written

$$\Delta r_{cl} = (\overline{r^2} - \bar{r}^2)^{1/2} = a\epsilon(1 - \epsilon^2/2)^{1/2}/\sqrt{2}. \quad (39)$$

However to facilitate the comparison of the classical and quantum results, we further substitute from Eqs. (32) and (33) to find that

$$\Delta r_{cl} = |K|\, (1/E^2 - 4L^4/m^2K^4)^{1/2}/4. \quad (40)$$

The quantum-mechanical result has effectively been provided by Wilhelm[14] who shows, with

$$a = h^2/4\pi^2 m|K|, \quad (41)$$

that

$$(\overline{r^2} - \bar{r}^2) = (a/2)^2[n^2(n^2 + 2) - l^2(l + 1)^2], \quad (42)$$

for a one-electron atom whose principal and angular momentum quantum numbers are n and l, respectively. Thus

$$\Delta r_q = (\overline{r^2} - \bar{r}^2)^{1/2} = (a/2)[n^2(n^2 + 2) - l^2(l + 1)^2]^{1/2}. \quad (43)$$

Or, upon substituting from Eq. (41), recalling that

$$E_n = -2\pi^2 mK^2/n^2h^2 \quad (44)$$

and

$$L^2 = l(l + 1)h^2/4\pi^2, \quad (45)$$

Δr_q can be rewritten

$$\Delta r_q = |K|(1/E_n^2 - 4L^4/m^2K^4 + h^2/\pi^2 mK^2|E_n|)^{1/2}/4. \quad (46)$$

In comparing Eqs. (40) and (46), one finds, for $E = E_n$, that in the classical limit of $h \to 0$, the quantum and classical results are identical. It is also observed, however, that Δr_q is otherwise greater than Δr_{cl}; additionally it is noted that Δr_q is always greater than zero, in contrast to the classical case where $\Delta r_{cl} = 0$ for $\epsilon = 0$, or $E = -mK^2/2L^2$, i.e., when the classical trajectory is a circle.

The classical expression for the radial momentum[13] is

$$p_r = \frac{L}{r^2}\frac{dr}{d\theta} = \frac{L\epsilon\sin\theta}{a(1 - \epsilon^2)}, \quad (47)$$

which, with Eq. (34), yields

$$\bar{p}_r = \int_0^\tau p_r dt \Big/ \int_0^\tau dt$$
$$= \frac{\epsilon L}{a(1-\epsilon^2)} \int_0^{2\pi} \sin\theta(1+\epsilon\cos\theta)^{-2}\, d\theta$$
$$\times \left(\int_0^{2\pi} (1+\epsilon\cos\theta)^{-2} d\theta \right)^{-1}. \quad (48)$$

However,

$$\int_0^{2\pi} \sin\theta(1+\epsilon\cos\theta)^{-2}\, d\theta = 0 \quad (A14)$$

and thus

$$\bar{p}_r = 0. \quad (49)$$

The evaluation of $\overline{p_r^2}$, which is also necessary for a determination of the momentum uncertainty, is completed by recalling[13] that the total energy E, of a particle moving under the influence of a inverse square force may be expressed as

$$E = p_r^2/2m + L^2/2mr^2 + K/r; \quad (50)$$

therefore upon solving for p_r^2, taking mean values, and recalling from the virial theorem that $(\overline{K/r}) = 2E$, we can write

$$\overline{p_r^2} = -2mE - L^2(\overline{1/r^2}). \quad (51)$$

Now upon employing Eqs. (31), (34), and (A7), one finds that

$$\overline{\frac{1}{r^2}} = \int_0^\tau r^{-2}(t)dt \Big/ \int_0^\tau dt$$
$$= a^{-2}(1-\epsilon^2)^{-2} \int_0^{2\pi} d\theta \Big/ \int_0^{2\pi} (1+\epsilon\cos\theta)^{-2} d\theta$$
$$= \frac{1}{a^2(1-\epsilon^2)^{1/2}}; \quad (52)$$

and if one substitutes from Eqs. (32) and (33), $(\overline{1/r^2})$ can be rewritten

$$(\overline{1/r^2}) = 2|E|(-2mE)^{1/2}/|K|L. \quad (53)$$

Thus

$$\overline{p_r^2} = -2mE - 2L|E|(-2mE)^{1/2}/|K| \quad (54)$$

and the classical radial momentum uncertainty is found to be

$$\Delta p_{r_{cl}} = (\overline{p_r^2} - \bar{p}_r^2)^{1/2}$$
$$= [-2mE - 2L|E|(-2mE)^{1/2}/|K|]^{1/2}. \quad (55)$$

Although it is noted that the radial momentum operator,

$$p_r = -i\frac{h}{2\pi}\left(\frac{\partial}{\partial r} + r^{-1}\right) \quad (56)$$

is, in general, not Hermitian,[15] we nonetheless calculate Δp_{r_q} in order to provide a comparison with the classical result. Since $\bar{p}_r = 0$,[16] we need only determine $\overline{p_r^2}$ which is most readily calculated from the quantum-mechanical analogue of Eq. (51), i.e.,

$$\overline{p_r^2} = -2mE_n - L^2(\overline{1/r^2}). \quad (57)$$

Since $(\overline{1/r^2})$ is known[17] to be

$$(\overline{1/r^2}) = 2/(2l+1)n^3a^2, \quad (58)$$

one finds, after substituting from Eqs. (41) and (44), that

$$\overline{p_r^2} = -2mE - 8\pi L^2|E_n|(-2mE_n)^{1/2}/(2l+1)h|K|. \quad (59)$$

Thus the quantum-mechanical radial momentum uncertainty is found to be

$$\Delta p_{r_q} = (\overline{p_r^2} - \bar{p}_r^2)^{1/2}$$
$$= (-2mE_n - 8\pi L^2|E_n|$$
$$\times(-2mE_n)^{1/2}/(2l+1)h|K|)^{1/2}. \quad (60)$$

Upon comparing $\Delta p_{r_{cl}}$ and Δp_{r_q}, it is indeed again found, in the classical limit, that the two expressions are identical for $E = E_n$. Namely, Eq. (60) reduces to Eq. (55) in the limit of very large values of the angular momentum quantum number where

$$\lim_{l\to\infty} (2l+1)h = 2lh = 4\pi L. \quad (61)$$

VI. CLASSICAL AND QUANTUM UNCERTAINTY PRINCIPLE

In each of the three examples considered above, it has been found that the quantum-mechanical position and momentum uncertainties retain finite values in the classical limit. It is thus not unreasonable to inquire into the existence of a classical uncertainty principle and its relationship to the quantum-mechanical result.

Let us begin by considering two variables A and B from which we define

$$D = \delta A + \lambda\delta B, \quad (62)$$

where λ is an arbitrary parameter and δA and δB are the deviations of A and B from their respective means, i.e.,

$$\delta A = A - \bar{A} \quad (63)$$

and

$$\delta B = B - \bar{B}. \quad (64)$$

It is observed that

$$\overline{D^2} \geq 0 \quad (65)$$

or

$$(\overline{\delta A^2 + 2\lambda\delta A\delta B + \lambda^2\delta B^2}) \geq 0. \quad (66)$$

However since

$$\overline{\delta A^2} = (\Delta A)^2, \quad (67)$$

$$\overline{\delta B^2} = (\Delta B)^2, \quad (68)$$

and

$$\overline{\delta A\delta B} = \overline{AB} - \bar{A}\,\bar{B}, \quad (69)$$

one finds, from Eq. (66), the following positive-definite quadratic relationship in λ,

$$(\Delta A)^2 + 2\lambda(\overline{AB} - \bar{A}\,\bar{B}) + \lambda^2(\Delta B)^2 \geq 0, \quad (70)$$

which implies that

$$(\Delta A)^2(\Delta B)^2 \geq (\overline{AB} - \bar{A}\,\bar{B})^2. \quad (71)$$

Or,

$$\Delta A\Delta B \geq |\overline{AB} - \bar{A}\,\bar{B}|: \quad (72)$$

the desired classical uncertainty principle where $\overline{AB} - \overline{A}\,\overline{B}$ is the covariance of A and B.[18] For $A = x$ and $B = p_x$, one finds

$$\overline{xp_x} - \overline{p}_x\overline{x} = 0, \tag{73}$$

and therefore

$$\Delta x \Delta p_x \geq 0. \tag{74}$$

Although Eqs. (1) and (74) are observed to be compatible for $h = 0$, it is nonetheless of interest to inquire into the relationship of Eq. (72) to the results of quantum mechanics. Let us thus examine the derivation of the Heisenberg principle.[19]

One begins by defining the kets $|\gamma\rangle$ and $|\beta\rangle$ from the Hermitian operators A and B:

$$|\gamma\rangle = (\mathrm{A} - \overline{A})|\phi\rangle \tag{75}$$

and

$$|\beta\rangle = (B - \overline{B})|\phi\rangle; \tag{76}$$

thus after noting that

$$\langle\gamma|\gamma\rangle = (\Delta A)^2, \tag{77}$$

$$\langle\beta|\beta\rangle = (\Delta B)^2, \tag{78}$$

and invoking the Schwarz inequality,

$$\langle\gamma|\gamma\rangle\langle\beta|\beta\rangle \geq |\langle\gamma|\beta\rangle|^2, \tag{79}$$

it is found that

$$(\Delta A)^2(\Delta B)^2 \geq \overline{C}^2/4 + \overline{D}^2, \tag{80}$$

where

$$C = (AB - BA)/i = [A,B]/i \tag{81}$$

and

$$D = [(A - \overline{A})(B - \overline{B}) + (B - \overline{B})(A - \overline{A})]/2. \tag{82}$$

One now restricts the argument to operators, A and B, for which

$$[A,B] = iR; \tag{83}$$

and after noting that $\overline{D}^2$ is positive definite, one finds

$$\Delta A \Delta B \geq |\overline{R}/2|, \tag{84}$$

which for $A = x$, $B = p_x$, and $R = h/2\pi$ indeed reduces to

$$\Delta x \Delta p_x \geq h/4\pi. \tag{85}$$

Although the above argument successfully rationalizes Eq. (1), it is overly restrictive when attempting to compare the classical uncertainty principle with the results of quantum mechanics. Namely, it is observed that

$$\overline{D} = (\overline{AB + BA})/2 - \overline{A}\,\overline{B} \tag{86}$$

is the Hermitized form of the classical covariance; and thus for operators obeying Eq. (83), we here write instead that

$$\Delta A \Delta B \geq \{[(\overline{AB + BA})/2 - \overline{A}\,\overline{B}]^2 + \overline{R}^2/4\}^{1/2}, \tag{87}$$

a form of the uncertainty principle first derived by Schrödinger.[20]

A comparison of the classical and quantum uncertainty principles, Eqs. (72) and (87), respectively, reveals that the latter differs from the former by the term $\overline{R}^2/4$; however it is also observed that the operators A and B commute in the classical limit and thus the quantum result does reduce to the classical form. Further, an interesting special case of Eq. (87) occurs for operators, such as $A = x^2$ and $B = p^2$, which are intrinsically real and are applied to systems with wavefunctions which are themselves real. Namely, one finds[21]

$$\overline{[A,B]} = 0 \tag{88}$$

and Eq. (87) then reduces to

$$\Delta A \Delta B \geq |(\overline{AB + BA})/2 - \overline{A}\,\overline{B}|, \tag{89}$$

which is essentially identical to the classical result.

Also of interest is a comparison of the physical implications of the quantum and classical uncertainty principles for position and momentum, i.e., Eqs. (1) and (74). In the usual interpretation,[1-4] one concludes, due to the finite lower bound on the product of Δx and Δp_x, that Eq. (1) prohibits an exact simultaneous determination of a particle's position and momentum. Presumably if this argument is applied to Eq. (74), it is found that classical physics does not include such a prohibition.

However if one considers Eqs. (1) and (74) in the context of the statistical interpretation, the conclusions are somewhat different. That is, Eq. (74) would seemingly now permit the preparation of an ensemble of classical particles for which Δx and/or Δp_x is zero. This implies the permissibility of finding a classical particle in the same position after a repeated number of observations, i.e., to observe it at rest. In contrast, Eq. (1) would now imply that neither Δx nor Δp_x can be found to be zero and thus a repeated number of observations of a quantum particle will find it occupying different positions at different times, i.e., it will never be observed at rest. This, as noted by several others,[22,23] seems to suggest a relationship between the quantum uncertainty principle and zero-point motion.

VII. SUMMARY

It has been demonstrated, in the classical limit, that the quantum position and momentum uncertainties of the particle in a box, the harmonic oscillator, and the one-electron atom approach those calculated from the classical probability density; additionally it has been suggested, in several cases, that the magnitude and nature of these uncertainties are consistent with a statistical interpretation. Further a classical uncertainty principle has been derived and compared with the quantum mechanical result; it has also been found, in the statistical interpretation, that there appears to be a relationship between position uncertainty and zero-point motion.

A question not entertained, but nonetheless of significance, is whether the statistical interpretation of Δx and Δp_x implies the Heisenberg interpretation, i.e., the existence of a finite limit on the precision of a single simultaneous observation of a particle's position and momentum. It is observed that Jammer[24] notes the possibility that the former does imply the latter.

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APPENDIX

Real integrals of the form

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta, \qquad \text{(A1)}$$

can be evaluated with the aid of Cauchy's residue theorem after making the substitutions[25]

$$d\theta = -idz/z, \qquad \text{(A2)}$$

$$\cos\theta = (z + 1/z)/2, \qquad \text{(A3)}$$

and

$$\sin\theta = (z - 1/z)/2i. \qquad \text{(A4)}$$

One finds that

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-2}d\theta = \frac{-4i}{\epsilon^2} \oint z\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-2} dz, \quad |z| = 1, \qquad \text{(A5)}$$

but since the integrand of Eq. (A5) has a second-order pole at $z = (-1 + \sqrt{1 - \epsilon^2})/\epsilon$,

$$\oint z\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-2} dz = 2\pi i Res\left[z\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-2}, \frac{-1 + \sqrt{1 - \epsilon^2}}{\epsilon}\right] = \frac{\pi i^2}{2(1 - \epsilon^2)^{3/2}}, \qquad \text{(A6)}$$

thus

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-2}d\theta = \frac{2\pi}{(1 - \epsilon^2)^{3/2}}. \qquad \text{(A7)}$$

Further,

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-3}\, d\theta = \frac{-8i}{\epsilon^3} \oint z^2\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-3} dz, \quad |z| = 1 \qquad \text{(A8)}$$

where

$$\oint z^2\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-3} dz = 2\pi i Res\left[z^2\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-3}, \frac{-1 + \sqrt{1 - \epsilon^2}}{\epsilon}\right] = \frac{\pi i(2 + \epsilon^2)\epsilon^3}{8(1 - \epsilon^2)^{5/2}}, \qquad \text{(A9)}$$

since there is a third-order pole at $z = (-1 + \sqrt{1 - \epsilon^2})/\epsilon$.

Therefore

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-3}\, d\theta = \frac{(2 + \epsilon^2)\pi}{(1 - \epsilon^2)^{5/2}}. \qquad \text{(A10)}$$

Also,

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-4}d\theta = (-16i\epsilon^4) \oint z^3\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-4} dz, \quad |z| = 1, \qquad \text{(A11)}$$

and

$$\oint z^3\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-4} dz = 2\pi i Res\left[z^3\left(z^2 + \frac{2}{\epsilon}z + 1\right)^{-4}, \frac{-1 + \sqrt{1 - \epsilon^2)}}{\epsilon}\right] = \frac{\pi i(2 + 3\epsilon^2)\epsilon^4}{16(1 - \epsilon^2)^{7/2}}, \qquad \text{(A12)}$$

since there is a fourth-order pole at $z = (-1 + \sqrt{1 - \epsilon^2})/\epsilon$.

Consequently,

$$\int_0^{2\pi} (1 + \epsilon\cos\theta)^{-4}\, d\theta = \frac{(2 + 3\epsilon^2)\pi}{(1 - \epsilon^2)^{7/2}}. \qquad \text{(A13)}$$

Finally,

$$\int_0^{2\pi} \sin\theta(1 + \epsilon\cos\theta)^{-2}d\theta = 0, \qquad \text{(A14)}$$

as may be readily observed after making the substitution

$$x = \cos\theta \qquad \text{(A15)}$$

and

$$d\theta = -dx(\sin\theta). \qquad \text{(A16)}$$

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[11]For a particle executing periodic motion between points x_1 and x_2, its average momentum is

$$\bar{p} = \left(\int_{x_1}^{x_2} (mv)v^{-1}dx - \int_{x_2}^{x_1} (-mv)v^{-1}dx\right) \times \left(\int_{x_1}^{x_2} v^{-1}\, dx - \int_{x_2}^{x_1} v^{-1}dx\right)^{-1} = 0.$$

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$$\bar{p}_r = -i\frac{h}{2\pi}\int_0^\infty R(r)\left(\frac{\partial R(r)}{\partial r} + r^{-1}R(r)\right) r^2 dr.$$

However upon integrating by parts, and noting that $R(0) = R(\infty) = 0$ for a one-electron atom, one also finds that

$$\bar{p}_r = i\frac{h}{2\pi}\int_0^\infty R(r)\left(\frac{\partial R(r)}{\partial r} + r^{-1}R(r)\right) r^2 dr,$$

which implies that $\bar{p}_r = 0$.
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